GLOBAL SMOOTH SOLUTIONS AND EXPONENTIAL STABILITY FOR A NONLINEAR BEAM*

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Abstract. In this paper we consider a dynamical system with boundary input and output describing the bending vibrations of a quasi-linear beam, where the nonlinearity comes from Hooke's law. First we derive an existence result for short-time solutions of the system of equations. Then we show that the structure of the boundary input and output forces the system to admit global solutions at least when the initial data and the boundary input are small in a certain sense. In particular, we prove that the norm of the state of the system decays exponentially if the input becomes zero after a finite time (the input being zero can be understood as a boundary feedback).

Key words. quasi-linear beam, exponential stabilization, energy-preserving system, boundary input and output

AMS subject classifications. 93C20, 35B35, 35B37, 35L65, 35L70

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1. Introduction and main results. We consider a system modeling the bending vibrations of a nonlinear beam of length $L > 0$. Let ϕ denote the deflection of the beam and let M be the bending torque. Assuming unit mass density, from Segel and Handelman [12] we have

$$\phi_{tt} - \kappa \phi_{tt}'' + M'' = 0, \quad x \in [0, L], \quad t > 0,$$

where $\kappa > 0$, $\sqrt{\kappa}$ is the radius of gyration, and the superscript “ $\prime$ ” and the subscript t denote differentiation with respect to the variables x (space) and t (time), respectively. We assume that the constitutive law (Hooke's law) is nonlinear, determined by a smooth function σ , such that $M = \sigma(\phi'')$. The beam equation then becomes

$$(1.1) \quad \phi_{tt} - \kappa \phi_{tt}'' + \sigma(\phi'')'' = 0, \quad x \in [0, L], \quad t > 0.$$

If $\sigma(x) = ax$, with $a > 0$, then the model is linear, called the *Rayleigh beam* in the literature. Recent papers dealing with the boundary stabilization of Rayleigh beams are Ammari, Liu, and Tucsnak [2], Guo, Wang, and Yung [7], and Weiss and Curtain [17].

Remark 1.1. We can derive (1.1) also from the calculus of variations. Let $T > 0$ be fixed. We define the *kinetic energy* of the beam at time $t \in [0, T]$ by $\mathbf{K}(\phi_t(\cdot, t))$, where the quadratic functional $\mathbf{K}$ is defined by

$$\mathbf{K}(\psi(\cdot, t)) = \frac{1}{2} \int_0^L [\psi^2(x, t) + \kappa \psi'^2(x, t)] dx.$$

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The *potential energy* of the beam at time t is defined as

$$\mathbf{P}(\phi(\cdot, t)) = \int_0^L F(\phi''(x, t)) dx,$$

where F is a smooth function such that $\frac{dF(z)}{dz} = \sigma(z)$. Then it is easy to check that the partial differential equation (1.1) is equivalent to the stationary point condition derived from Hamilton's principle

$$\left. \frac{d}{ds} \int_0^T \mathbf{L}(\phi + sv, \phi_t + sv_t) dt \right|_{s=0} = 0 \quad \forall v \in C_0^\infty(Q_T),$$

where $\mathbf{L}(\phi, \psi) = \mathbf{K}(\psi) - \mathbf{P}(\phi)$ and $Q_T = (0, L) \times (0, T)$. Indeed, this follows from

$$\left\langle \frac{\partial \mathbf{L}}{\partial \phi}(\phi, \psi), v \right\rangle = -\langle \sigma(\phi''), v'' \rangle, \quad \left\langle \frac{\partial \mathbf{L}}{\partial \psi}(\phi, \psi), v_t \right\rangle = \langle \psi, v_t \rangle + \kappa \langle \psi', v'_t \rangle,$$

where all the inner products are taken in $L^2(Q_T)$.

Suppose that the initial data are given by

$$(1.2) \quad \phi(x, 0) = \phi^0(x), \quad \phi_t(x, 0) = \phi^1(x), \quad x \in [0, L].$$

We suppose that the right end of the beam is hinged, i.e.,

$$(1.3) \quad \phi(L, t) = \phi''(L, t) = 0,$$

and at the left end we have actuators and sensors, linked in a damping feedback mechanism, satisfying the equations

$$(1.4) \quad \begin{cases} 2b_1 u_1(t) = \sigma(\phi''(0, t)) - b_1^2 \phi'_t(0, t), & t \geq 0, \\ 2b_2 u_2(t) = (\sigma(\phi''))'(0, t) - \kappa \phi'_{tt}(0, t) + b_2^2 \phi_t(0, t), & t \geq 0, \end{cases}$$

$$(1.5) \quad \begin{cases} 2b_1 y_1(t) = \sigma(\phi''(0, t)) + b_1^2 \phi'_t(0, t), & t \geq 0, \\ 2b_2 y_2(t) = (\sigma(\phi''))'(0, t) - \kappa \phi'_{tt}(0, t) - b_2^2 \phi_t(0, t), & t \geq 0, \end{cases}$$

where $b_i > 0$ are constants for $i = 1, 2$, $u = (u_1, u_2)$ is the input function, and $y = (y_1, y_2)$ is the output function. The term $2b_1 u_1(t)$ is an external torque acting at the left end of the beam, while $2b_2 u_2(t)$ is an external transversal force acting at the same end. The term $\sigma(\phi''(0, t))$ is the total torque acting at $x = 0$, while $(\sigma(\phi''))'(0, t) - \kappa \phi'_{tt}(0, t)$ is the total transversal force at $x = 0$. The terms $-b_1^2 \phi'_t(0, t)$ and $b_2^2 \phi_t(0, t)$ represent the boundary feedback torque and force, respectively.

To guarantee that the system (1.1)–(1.5) is hyperbolic, we further assume that

$$(1.6) \quad \sigma(0) = 0, \quad \sigma'(z) > 0, \quad z \in (-\infty, \infty).$$

If we define the *total energy* of the system (1.1)–(1.5) by

$$\mathbf{E}(t) = \mathbf{K}(\phi_t(\cdot, t)) + \mathbf{P}(\phi(\cdot, t)),$$

where $\mathbf{K}$ and $\mathbf{P}$ are defined as in Remark 1.1, we obtain by a simple computation

$$(1.7) \quad \begin{aligned} \frac{d\mathbf{E}(t)}{dt} &= -\sigma(\phi''(0, t))\phi'_t(0, t) + [\sigma(\phi'')'(0, t) - \kappa \phi'_{tt}(0, t)]\phi_t(0, t) \\ &= |u(t)|^2 - |y(t)|^2. \end{aligned}$$

This formula expresses that the rate of change of the energy is equal to the power supplied to the system by the input minus the power taken out by the output. The balance equation (1.7) means that the system (1.1)–(1.5) is (scattering) *energy-preserving*. We refer to Staffans [14] and to Malinen, Staffans, and Weiss [11] for the theory of energy-preserving linear systems.

We define the positive smooth function $\mathcal{N} : (\sigma_-, \sigma_+) \rightarrow (0, \infty)$ by

$$(1.8) \quad \mathcal{N}(z) = \sigma'(\sigma^{-1}(z)),$$

where σ^{-1} is the inverse function of σ and

$$(1.9) \quad \sigma_{\pm} = \lim_{z \rightarrow \pm\infty} \sigma(z).$$

By the assumption (1.6), the above limits exist (they may be infinite) and

$$(1.10) \quad 0 \in (\sigma_-, \sigma_+), \quad \lim_{z \rightarrow \sigma_{\pm}} \sigma^{-1}(z) = \pm\infty.$$

In the linear case ($\sigma(z) = az$) we have $\mathcal{N}(z) = a$, of course.

We introduce an equivalent system by the change of unknowns

$$\varphi = \sigma(\phi''), \quad \psi = \phi_t.$$

Then the system (1.1)–(1.5) becomes

$$(1.11) \quad \begin{cases} \varphi_t = \mathcal{N}(\varphi)\psi'', \\ \psi_t = \kappa\psi_t'' - \varphi'', \\ \varphi(x, 0) = \varphi_0(x), \quad \psi(x, 0) = \psi_0(x), \\ \varphi(L, t) = \psi(L, t) = \psi''(L, t) = 0, \end{cases}$$

with the input signal $u = (u_1, u_2)$ acting through

$$(1.12) \quad \begin{cases} 2b_1u_1(t) = \varphi(0, t) - b_1^2\psi'(0, t), \\ 2b_2u_2(t) = \varphi'(0, t) - \kappa\psi_t'(0, t) + b_2^2\psi(0, t), \end{cases}$$

and with the output signal $y = (y_1, y_2)$ defined by

$$(1.13) \quad \begin{cases} 2b_1y_1(t) = \varphi(0, t) + b_1^2\psi'(0, t), \\ 2b_2y_2(t) = \varphi'(0, t) - \kappa\psi_t'(0, t) - b_2^2\psi(0, t). \end{cases}$$

The initial functions of this new system are obtained from those in (1.2) by

$$\varphi_0 = \sigma(\phi^{0''}), \quad \psi_0 = \phi^1.$$

It is well known that smooth solutions of quasi-linear hyperbolic systems usually develop singularities after some time; see, for example, Klainerman and Majda [10]. However, since the structure of the boundary conditions (1.12) causes energy dissipation (as seen in (1.7)), we expect that the introduction of the boundary input (1.12) prevents the formation of singularities and assures the existence of a global smooth solution. In fact, such problems have been considered for the nonlinear string in Greenberg and Li [6], Slemrod [13], Alber and Cooper [1], and Yao [18].

The main aim of this paper is to study the existence of smooth short-time solutions and smooth global solutions of the nonlinear beam system (1.11)–(1.12) if the initial data φ_0, ψ_0 and the boundary input u are given. For short-time solutions, we use the results of Kato [8] to establish the existence of solutions of a certain linear system and then use an argument with contraction mappings from Dafermos and Hrusa [4] to obtain the existence. Due to the special structure of (1.12), some modifications are necessary in the energy estimates which are used in the fixed-point argument. For global solutions, we establish an estimate for the boundary trace by invoking some ideas from Alber and Cooper [1] and from Cirina [3]. Such an estimate is necessary for the global existence. In the linear case (where $\sigma(x) = ax$), boundary trace estimates are usually derived by the multiplier technique.

We mention that in the linear case our system fits into the framework of Weiss and Tucsnak [16] and [15], which guarantees that the system is well-posed and conservative (this property is stronger than “energy-preserving”).

If $b_2 = 0$, the system (1.11)–(1.13) can be transformed into a quasi-linear string with an input-output structure at the boundary that has been studied in [18] (see Remark 4.2). Here we study only the case $b_2 > 0$.

Motivated by the structure of the function $\mathcal{N}$, we introduce the following concept. We say that a continuous function φ on $[0, L]$ satisfies the σ -condition if

$$(1.14) \quad \sigma_- < \inf_{x \in [0, L]} \varphi(x) \leq \sup_{x \in [0, L]} \varphi(x) < \sigma_+,$$

where σ_-, σ_+ are as in (1.9). If $\sigma_{\pm} = \pm\infty$, then this condition is trivially true.

For short-time solutions, we have the following existence result.

THEOREM 1.1. *Suppose that $u = (u_1, u_2) \in C^3[0, \infty) \times C^2[0, \infty)$ and $(\varphi_0, \psi_0) \in H^2(0, L) \times H^3(0, L)$ satisfies the boundary conditions*

$$(1.15) \quad \varphi_0(L) = \psi_0(L) = \psi_0''(L) = 0$$

and the compatibility conditions

$$(1.16) \quad \begin{cases} 2b_1 u_1(0) = \varphi_0(0) - b_1^2 \psi_0'(0), \\ 2b_2 u_2(0) = 2b_1^{-1} \kappa u_{1t}(0) + \varphi_0'(0) - \kappa b_1^{-2} \mathcal{N}(\varphi_0(0)) \psi_0''(0) + b_2^2 \psi_0(0). \end{cases}$$

Furthermore, we assume that φ_0 satisfies the σ -condition (1.14).

Then there is a $T_0 > 0$ such that the problem (1.11)–(1.12) admits a unique solution (φ, ψ) on $Q_{T_0} = (0, L) \times (0, T_0)$ satisfying

$$(1.17) \quad \varphi \in \bigcap_{i=0}^2 C^i([0, T_0]; H^{2-i}(0, L)), \quad \psi \in \bigcap_{i=0}^3 C^i([0, T_0]; H^{3-i}(0, L)),$$

where $H^0(0, L)$ stands for $L^2(0, L)$.

The proof of this theorem is given in section 3.

When the input u and the initial data (φ_0, ψ_0) are appropriately small, the solutions are global. To describe this, we introduce an energy-like function $\mathcal{E}$ as follows. Let (φ, ψ) be a solution of the system (1.11)–(1.12). For any multi-index $\alpha = (\alpha_1, \alpha_2) \in \{0, 1, 2, \dots\}^2$, we denote $|\alpha| = \alpha_1 + \alpha_2$. Define

$$(1.18) \quad \mathcal{E}(t) = [\varphi'(0, t)]^2 + \|\psi'_{tt}(\cdot, t)\|^2 + \sum_{|\alpha| \leq 2} (\|\partial^\alpha \varphi(\cdot, t)\|^2 + \|\partial^\alpha \psi(\cdot, t)\|^2),$$

where $\partial^{(\alpha_1, \alpha_2)} = \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial t}\right)^{\alpha_2}$ and $\|\cdot\|$ is the usual norm on $L^2(0, L)$.

For $u = (u_1, u_2) \in [H^2(0, T)]^2$, denote

$$(1.19) \quad \|u\|_{H^2(0, T)}^2 = \int_0^T |u|_2^2(t) dt, \quad |u|_2^2(t) = |u(t)|^2 + |u_t(t)|^2 + |u_{tt}(t)|^2.$$

THEOREM 1.2. *Suppose that $u = (u_1, u_2) \in H^2(0, \infty) \times H^2(0, \infty)$ and the initial data $(\varphi_0, \psi_0) \in H^2(0, L) \times H^3(0, L)$ are such that the boundary conditions (1.15) and the compatibility conditions (1.16) hold. Then there are $\epsilon > 0$, $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, independent of (φ_0, ψ_0) and u , such that if*

$$(1.20) \quad \mathcal{E}(0) + \|u\|_{H^2(0, \infty)}^2 \leq \epsilon,$$

then the problem (1.11)–(1.12) has a unique solution (φ, ψ) on $Q_\infty = (0, L) \times (0, \infty)$ and

$$(1.21) \quad \mathcal{E}(t) + c_1 \int_0^t \mathcal{E}(\tau) d\tau \leq c_2 \mathcal{E}(0) + c_3 \|u\|_{H^2(0, \infty)}^2 \quad \forall t \geq 0.$$

Note that from (φ_0, ψ_0) we can compute $\partial^\alpha \varphi(\cdot, 0)$ and $\partial^\alpha \psi(\cdot, 0)$ by solving (1.11) at $t = 0$, so that also $\mathcal{E}(0)$ can be computed.

The above theorem will be proved in section 5.

COROLLARY 1.1. *Under the assumptions of Theorem 1.2, if (1.20) holds, then the solution (φ, ψ) of (1.11)–(1.12) and the output $y = (y_1, y_2)$ from (1.13) satisfy*

$$(1.22) \quad \varphi \in H^2(Q_\infty), \quad \psi \in H^2(Q_\infty) \cap H^{1,2}(Q_\infty), \quad y \in H^2(0, \infty) \times H^2(0, \infty),$$

where $H^{1,2}(Q_\infty)$ is the Sobolev space with the norm

$$\|\psi\|_{H^{1,2}(Q_\infty)}^2 = \sum_{0 \leq i \leq 1; 0 \leq j \leq 2} \int_0^\infty \|\partial^{(i,j)} \psi(\cdot, t)\|^2 dt.$$

The following stability result will also be proved in section 5.

THEOREM 1.3. *Under the assumptions of Theorem 1.2, suppose that (φ_0, ψ_0) with u satisfy (1.20). If there is a $T_0 > 0$ such that*

$$(1.23) \quad u(t) = 0 \quad \forall t \geq T_0,$$

then there are $C > 0$ and $\epsilon_1 > 0$ such that

$$(1.24) \quad \mathcal{E}(t) \leq C e^{-\epsilon_1 t} \quad \forall t \geq 0.$$

This theorem shows that the energy-like function $\mathcal{E}(t)$ for the boundary feedback system corresponding to $u = 0$ decays exponentially.

REMARK 1.2. We do not assume that φ_0 satisfies the σ -condition in Theorems 1.2, 1.3 and in Corollary 1.1, because for suitable ε , the small initial data assumption (1.20) implies this. Indeed,

$$\sup_{x \in [0, L]} |\varphi_0(x)| \leq L^{\frac{1}{2}} \mathcal{E}^{\frac{1}{2}}(0) \leq L^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}.$$

Then the σ -condition is true if $0 < \varepsilon < L^{-1} \min\{\sigma_-^2, \sigma_+^2\}$.

2. Linear time-varying equations. Let $T > 0$ and let N be a function on $Q_T = (0, L) \times (0, T)$ such that

$$N \in \bigcap_{i=0}^2 C^i([0, T]; H^{2-i}(0, L))$$

and there are $N_0 > 0$, $N_1 > 0$ such that

$$(2.1) \quad N_0 \leq N(x, t) \leq N_1, \quad (x, t) \in Q_T.$$

Motivated by (1.11)–(1.12), we consider the linear dynamical system described by

$$(2.2) \quad \begin{cases} \varphi_t = N\psi'', \\ \psi_t = \kappa\psi_t'' - \varphi'', \\ \varphi(x, 0) = \varphi_0(x), \quad \psi(x, 0) = \psi_0(x), \\ \varphi(L, t) = \psi(L, t) = \psi''(L, t) = 0, \\ 2b_1u_1(t) = \varphi(0, t) - b_1^2\psi'(0, t), \\ 2b_2u_2(t) = \varphi'(0, t) - \kappa\psi_t'(0, t) + b_2^2\psi(0, t). \end{cases}$$

We make a change of unknowns by

$$(2.3) \quad v = \varphi, \quad w = \psi + g \quad \text{on } Q_T,$$

where g is a smooth function on Q_T satisfying the boundary conditions

$$(2.4) \quad g(L, t) = 0, \quad b_1g'(0, t) = 2u_1(t), \quad t \in [0, T].$$

We shall specify later how to choose g in order to satisfy additional requirements. By (2.3)–(2.4), the system (2.2) becomes, for $(x, t) \in Q_T$,

$$(2.5) \quad \begin{cases} v_t - Nw'' = -Ng'', \\ w_t - \kappa w_t'' + v'' = g_t - \kappa g_t'', \\ v(x, 0) = \varphi_0(x), \quad w(x, 0) = \psi_0(x) + g(x, 0), \\ v(L, t) = w(L, t), \quad w''(L, t) = g''(L, t), \\ v(0, t) - b_1^2w'(0, t) = 0, \\ v'(0, t) - \kappa w_t'(0, t) + b_2^2w(0, t) = 2b_2u_2(t) + b_2^2g(0, t) - \kappa g_t'(0, t). \end{cases}$$

Let us formulate the problem (2.5) in an appropriate Hilbert space. First we introduce the following space:

$$H = \{f \mid f \in H^1(0, L), f(L) = 0\}, \quad \langle f, p \rangle_H = \langle f, p \rangle + \kappa \langle f', p' \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product of $L^2(0, L)$. To rewrite the second equation in (2.5), we introduce an operator $B_0: L^2(0, L) \rightarrow H$ by

$$(2.6) \quad \langle p, f' \rangle = \langle B_0p, f \rangle_H \quad \forall p \in L^2(0, L), \quad f \in H.$$

LEMMA 2.1. *The operator B_0 from (2.6) is an isomorphism between $L^2(0, L)$ and H . In addition, for every $p \in L^2(0, L)$,*

$$(2.7) \quad (B_0p)(x) = \langle p, q_0 \rangle \cosh \sqrt{a}x + a \int_0^x p(s) \cosh \sqrt{a}(x-s) ds,$$

where

$$(2.8) \quad a = \frac{1}{\kappa}, \quad q_0(x) = -\frac{a \cosh \sqrt{a}(L-x)}{\cosh \sqrt{a}L}.$$

In particular, $\frac{\partial}{\partial x} B_0$ is bounded from $L^2(0, L)$ to itself, and for $p \in L^2(0, L)$

$$(2.9) \quad \left(\frac{\partial}{\partial x} B_0 p \right) (x) = \sqrt{a} \langle p, q_0 \rangle \sinh \sqrt{a}x + a^{\frac{3}{2}} \int_0^x p(s) \sinh \sqrt{a}(x-s) ds + ap(x).$$

Proof. First, we assume that $p \in H^1(0, L)$. Denote $y = B_0 p$. From (2.6), integration by parts gives the problem

$$y - \kappa y'' = -p', \quad y'(0) = ap(0), \quad y(L) = 0.$$

Solving this problem, we obtain (2.7) with (2.8). Since $H^1(0, L)$ is dense in $L^2(0, L)$, (2.7) is actually true for all $p \in L^2(0, L)$. Differentiating (2.7) we obtain (2.9).

It is easy to check that B_0 is invertible and that $B_0^{-1}: H \rightarrow L^2(0, L)$ is given by

$$(B_0^{-1}f)(x) = \kappa f'(x) - \int_0^x f(s) ds \quad \forall f \in H. \quad \square$$

Let (v, w) be a smooth solution of the system (2.5). We multiply the second equation in (2.5) by a function $f \in H$ and integrate over $[0, L]$. Using integration by parts and (2.6) we obtain that for all $f \in H$,

$$\langle w_t, f \rangle_H = \langle B_0 v' + g_t, f \rangle_H + [v'(0, t)(t) - \kappa w_t'(0, t) + \kappa g_t'(0, t)] f(0).$$

Using the last equation in (2.5), this becomes

$$(2.10) \quad \langle w_t, f \rangle_H = \langle B_0 v' + g_t, f \rangle_H + [2b_2 u_2(t) - b_2^2 w(0, t) + b_2^2 g(0, t)] f(0).$$

Let $q_1 \in H$ be such that

$$(2.11) \quad f(0) = \langle q_1, f \rangle_H \quad \forall f \in H.$$

It follows from (2.10) and (2.11) that

$$w_t = B_0 v' - b_2^2 w(0, t) q_1 + g_t + [2b_2 u_2(t) + b_2^2 g(0, t)] q_1 \quad \text{in } H.$$

Now the problem (2.5) becomes a problem on $L^2(0, L) \times H$:

$$(2.12) \quad \begin{cases} v_t = Nw'' - Ng'', \\ w_t = B_0 v' - b_2^2 w(0, t) q_1 + g_t + [2b_2 u_2(t) + b_2^2 g(0, t)] q_1, \\ v(x, 0) = \varphi_0(x), \quad w(x, 0) = \psi_0(x) + g(x, 0), \\ v(L, t) = w(L, t) = 0, \quad v(0, t) = b_1^2 w'(0, t). \end{cases}$$

We wish to solve (2.12) by applying the results of Kato [8, 9] in the space $\mathcal{H} = L^2(0, L) \times H$, with the inner product

$$\left\langle \begin{bmatrix} p \\ f \end{bmatrix}, \begin{bmatrix} \hat{p} \\ \hat{f} \end{bmatrix} \right\rangle_{\mathcal{H}} = \langle p, \hat{p} \rangle + \langle f, \hat{f} \rangle_H \quad \forall \begin{bmatrix} p \\ f \end{bmatrix}, \begin{bmatrix} \hat{p} \\ \hat{f} \end{bmatrix} \in \mathcal{H}.$$

To this end, we introduce a family of closed operators on $\mathcal{H}$ by

$$(2.13) \quad \mathcal{A}_0(t) = \begin{bmatrix} 0 & N\partial^2 \\ B_0\partial & -b_2^2 q_1 \Gamma_0 \end{bmatrix}, \quad 0 \leq t \leq T,$$

where $\partial = \frac{\partial}{\partial x}$, the linear functional Γ_0 is given by

$$\Gamma_0 f = f(0) \quad \forall f \in H,$$

and the domain of $\mathcal{A}_0(t)$ is

$$(2.14) \quad D(\mathcal{A}_0(t)) = \left\{ \begin{bmatrix} p \\ f \end{bmatrix} \in H^1(0, L) \times H^2(0, L) \mid \begin{array}{l} p(L) = f(L) = 0, \\ p(0) = b_1^2 f'(0) \end{array} \right\}.$$

Then the problem (2.12) can be written as an evolution equation on $\mathcal{H}$:

$$(2.15) \quad X_t(t) = \mathcal{A}_0(t)X(t) + \mathcal{F}_0(t), \quad X(0) = X_0,$$

with

$$(2.16) \quad X = \begin{bmatrix} v \\ w \end{bmatrix}, \quad \mathcal{F}_0(t) = \begin{bmatrix} -Ng'' \\ g_t + [2b_2 u_2(t) + b_2^2 g(0, t)]q_1 \end{bmatrix}.$$

More precisely, the connection between (2.12) and (2.15) is as follows: Let

$$(2.17) \quad \mathcal{H}_1 = D(\mathcal{A}_0(t)),$$

which is a Hilbert space with the graph norm. This space is independent of t and for different t we get on it equivalent norms.

LEMMA 2.2. *If $X(\cdot) \in C^1([0, l]; \mathcal{H}_1)$ is a solution of (2.15), then its components (v, w) are a solution of (2.12).*

The proof of this lemma is by simple verification.

Note that $X(\cdot) \in C^1([0, T]; \mathcal{H}_1)$ implies $X(t) \in H^1(0, L) \times H^3(0, L)$ for all $t \in [0, T]$. For less regular solutions of (2.15), some of the equations in (2.12) become meaningless. For example, if we only have $X(\cdot) \in C([0, T]; \mathcal{H}_1)$, which is still rather smooth, then the last equation in (2.12) becomes meaningless.

Since the domain $D(\mathcal{A}_0(t))$ is independent of t (see (2.14)), the key of applying the work of Kato to our problem (2.15) is to show that $\{\mathcal{A}_0(t) \mid 0 \leq t \leq T\}$ is a stable family on $\mathcal{H}$, in the sense of [8]. We recall that $\{\mathcal{A}_0(t) \mid 0 \leq t \leq T\}$ is called a *stable family* if the following conditions hold:

- (1) For each $t \in [0, T]$, $\mathcal{A}_0(t)$ is the generator of a C_0 -semigroup.
- (2) There are constants M, β (called the constants of stability) such that

$$\left\| \prod_{j=1}^k (\lambda - \mathcal{A}_0(t_j))^{-1} \right\| \leq M(\lambda - \beta)^{-k}, \quad \lambda > \beta,$$

for any finite family $\{t_j\}$ with $0 \leq t_1 \leq \dots \leq t_k \leq T$, $k \in \mathbb{N}$. Here, the product Π is time-ordered; i.e., a factor with larger t_j stands to the left of one with smaller t_j .

LEMMA 2.3. *Let $\mathcal{A}_0(t)$ be defined in (2.13) and (2.14). For each t , $0 \leq t \leq T$, $\mathcal{A}_0(t)$ generates a C_0 -semigroup on $\mathcal{H}$.*

Proof. For $t \in [0, T]$ we introduce a family of new inner products $\langle \cdot, \cdot \rangle_t$ on $\mathcal{H}$ which are equivalent to the old one, to show that $\mathcal{A}_0(t)$ is dissipative on $\mathcal{H}$:

$$(2.18) \quad \left\langle \begin{bmatrix} p \\ f \end{bmatrix}, \begin{bmatrix} \hat{p} \\ \hat{f} \end{bmatrix} \right\rangle_t = \langle N^{-1}(\cdot, t)p, \hat{p} \rangle + \langle f, \hat{f} \rangle_H \quad \forall \begin{bmatrix} p \\ f \end{bmatrix}, \begin{bmatrix} \hat{p} \\ \hat{f} \end{bmatrix} \in \mathcal{H}.$$

Since N satisfies (2.1), these new norms are equivalent to the old one.

Let $\begin{bmatrix} p \\ f \end{bmatrix} \in D(\mathcal{A}_0(t))$. It follows from (2.13), (2.14), and (2.18) that

$$\begin{aligned} \left\langle \mathcal{A}_0(t) \begin{bmatrix} p \\ f \end{bmatrix}, \begin{bmatrix} p \\ f \end{bmatrix} \right\rangle_t &= \left\langle \begin{bmatrix} Nf'' \\ B_0p' - b_2^2f(0)q_1 \end{bmatrix}, \begin{bmatrix} p \\ f \end{bmatrix} \right\rangle_t \\ &= \langle f'', p \rangle + \langle p', f' \rangle - b_2^2f^2(0) = pf'|_0^L - b_2^2f^2(0) \\ &= -b_1^2f'^2(0) - b_2^2f^2(0) \leq 0. \end{aligned}$$

We want to show that $\mathcal{A}_0(t)$ is m-dissipative. For this, it is enough to show that also $\mathcal{A}_0^*(t)$ is dissipative, where $\mathcal{A}_0^*(t)$ is the adjoint operator of $\mathcal{A}_0(t)$ with respect to the inner product $\langle \cdot, \cdot \rangle_t$. A simple computation shows that

$$\mathcal{A}_0^*(t) = - \begin{bmatrix} 0 & N\partial^2 \\ B_0\partial & b_2^2q_1\Gamma_0 \end{bmatrix},$$

$$D(\mathcal{A}_0^*(t)) = \left\{ \begin{bmatrix} p \\ f \end{bmatrix} \in H^1(0, L) \times H^2(0, L) \mid \begin{array}{l} p(L) = f(L) = 0, \\ p(0) = -b_1^2f'(0) \end{array} \right\}$$

and for all $\begin{bmatrix} p \\ f \end{bmatrix} \in D(\mathcal{A}_0^*(t))$,

$$\left\langle \mathcal{A}_0^*(t) \begin{bmatrix} p \\ f \end{bmatrix}, \begin{bmatrix} p \\ f \end{bmatrix} \right\rangle_t = -b_1^2f'^2(0) - b_2^2f^2(0) \leq 0.$$

Thus, $\mathcal{A}_0(t)$ is m-dissipative. According to the Lumer–Phillips theorem, $\mathcal{A}_0(t)$ is the generator of a contraction semigroup on $\mathcal{H}$ (with respect to $\langle \cdot, \cdot \rangle_t$). $\square$

LEMMA 2.4. *For any $t \in [0, T]$, $\{\mathcal{A}_0(\tau) \mid 0 \leq \tau \leq T\}$ is a stable family with respect to the norm $\|\cdot\|_t$ (which corresponds to the inner product (2.18)) on $\mathcal{H}$.*

Proof. In view of Proposition 3.4 in [8] and Lemma 2.3 above, it suffices to verify that there exists $c > 0$ such that the following inequality holds:

$$(2.19) \quad \|\mathcal{X}\|_t / \|\mathcal{X}\|_s \leq e^{c|t-s|} \quad \forall \mathcal{X} = \begin{bmatrix} p \\ f \end{bmatrix} \in \mathcal{H}, \quad s, t \in [0, T].$$

To this end, we take

$$c \geq \sup_{(x,t) \in Q_T} |N_t|N^{-1}$$

and set

$$h(t) = \left\| \begin{bmatrix} p \\ f \end{bmatrix} \right\|_t^2 = \langle N^{-1}(\cdot, t)p, p \rangle + \|f\|_H^2, \quad t \in [0, T].$$

Then

$$|h_t(t)| \leq 2ch(t) \quad \forall \begin{bmatrix} p \\ f \end{bmatrix} \in \mathcal{H}, \quad t \in [0, T].$$

This inequality implies that the function $h(t)e^{-2ct}$ is decreasing while $h(t)e^{2ct}$ is increasing for $t \in [0, T]$. Therefore, (2.19) is true. $\square$

In order to have a solution $X(\cdot) \in C([0, T]; \mathcal{H}_1)$ to the inhomogeneous system (2.15), it will suffice to choose $X_0 \in \mathcal{H}_1$ and a function g such that the inhomogeneous term $\mathcal{F}_0(\cdot) \in C([0, T]; \mathcal{H}_1)$. For this, we consider the 3×3 matrix $\mathcal{B}$ defined by

$$\mathcal{B} = (e_2, e_3, e_1),$$

where (e_1, e_2, e_3) is the standard basis of the Euclidian space $\mathbb{R}^3$, and we take a function $\rho \in C^\infty[0, L]$ such that

$$(2.20) \quad \rho(x) = 1 \text{ for } x \leq L/3, \quad \rho(x) = 0 \text{ for } x \geq 2L/3.$$

LEMMA 2.5. *Assume that $u = (u_1, u_2) \in C^2[0, T] \times C^1[0, T]$. Using the notation $\mathcal{B}$ and ρ introduced above, there exists a function g of the form*

$$(2.21) \quad g(x, t) = \rho(x)\beta(t) \cdot e^{\mathcal{B}x}e_1, \quad (x, t) \in Q_T,$$

where $\beta(t) = [\beta_1(t) \ \beta_2(t) \ \beta_3(t)]$, such that g satisfies (2.4) and $\mathcal{F}_0(\cdot) \in C([0, T]; \mathcal{H}_1)$, where $\mathcal{F}_0(t)$ is given in (2.16).

Proof. It follows from (2.20)–(2.21) that

$$\begin{cases} g(L, t) = g''(L, t) = 0, \\ g(0, t) = \beta_1(t), \quad g'(0, t) = \beta_2(t), \quad g''(0, t) = \beta_3(t). \end{cases}$$

On the other hand, from (2.14) and (2.16) we see that the conditions (2.4) and $\mathcal{F}_0(t) \in D(\mathcal{A}_0(t))$ for $t \in [0, T]$ will be satisfied if

$$\begin{cases} g(L, t) = g''(L, t) = 0, & b_1 g'(0, t) = 2u_1(t), \\ N(0, t)g''(0, t) + q'_1(0)b_1^2b_2^2g(0, t) = -2b_1u_{1t}(t) - 2q'_1(0)b_1^2b_2u_2(t). \end{cases}$$

This can be achieved if we set

$$\begin{cases} \beta_1(t) = 0, & \beta_2(t) = 2b_1^{-1}u_1(t), \\ \beta_3(t) = -2b_1N^{-1}(0, t)[u_{1t}(t) + q'_1(0)b_1b_2u_2(t)], \end{cases}$$

and moreover we then have $\mathcal{F}_0(\cdot) \in C([0, T]; \mathcal{H}_1)$. $\square$

Since $D(\mathcal{A}_0(t))$ is constant, by Theorem 7.1 of [8] (see also Remark 6.2 of the same paper), for any $X_0 \in \mathcal{H}_1$ and for $\mathcal{F}_0$ as in the last lemma, the problem (2.15) admits a unique solution $X(\cdot) \in C([0, T]; \mathcal{H}_1)$. However, from here we cannot go back to the problem (2.2) through the change of unknowns (2.3), because the functions do not have enough smoothness to make sense of the boundary conditions.

For our purposes in this paper, we need the problem (2.2) to have solutions (φ, ψ) such that $\varphi \in C([0, T]; H^2(0, L))$ and $\psi \in C([0, T]; H^3(0, L))$. For this, we consider an evolutionary problem, like (2.15), on the space $(\mathcal{H}_1, \|\cdot\|_1)$ where $\mathcal{A}_0(t) : D(\mathcal{A}_0^2(t)) \rightarrow \mathcal{H}_1$ and $\mathcal{H}_1 = D(\mathcal{A}_0(t))$ with the graph norm. Using Lemma 2.1, a simple computation shows that $D(\mathcal{A}_0^2(t))$ consists of all the pairs $(v, w) \in H^2(0, L) \times H^3(0, L)$ satisfying the boundary conditions

$$v(L) = w(L) = w''(L) = 0, \quad v(0) = b_1^2w'(0),$$

$$N(0, t)w''(0) = ab_1^2v'(0) + ab_1^2b_2^2w(0).$$

The last boundary condition shows that $D(\mathcal{A}_0^2(t))$ depends on the time t . Thus, we would meet difficulty in verifying the assumptions of [8], [9], although the work of Kato does not assume that the domain is independent of time t .

We overcome the above difficulty as follows. Suppose that we have a solution X of the problem (2.15) such that

$$(2.22) \quad X(\cdot) \in C([0, T]; D(\mathcal{A}_0^2(t))).$$

After making a change of unknowns by

$$Y(t) = \mathcal{C}(t)X(t), \quad \mathcal{C}(t) = \lambda I - \mathcal{A}_0(t),$$

where λ is large enough, we observe that $Y(\cdot) \in C([0, T]; \mathcal{H}_1)$ satisfies

$$(2.23) \quad Y_t(t) = [\mathcal{A}_0(t) + \mathcal{C}_t(t)\mathcal{C}^{-1}(t)]Y(t) + \mathcal{C}(t)\mathcal{F}_0(t), \quad Y(0) = \mathcal{C}(0)X_0 \in \mathcal{H}_1.$$

Since the above argument is reversible, in order to obtain a solution to the problem (2.15) to satisfy (2.22), we need only solve the problem (2.23) in the space $\mathcal{H}$. This is achieved by the following two lemmas.

LEMMA 2.6. *For $t \in [0, T]$, $\mathcal{C}_t(t)\mathcal{C}^{-1}(t)$ is bounded from $\mathcal{H}$ to $\mathcal{H}$. Furthermore, for any $X \in \mathcal{H}$, $\mathcal{C}_t(\cdot)\mathcal{C}^{-1}(\cdot)X \in C([0, T]; \mathcal{H})$.*

Proof. From the formula (2.13), we have

$$(2.24) \quad \mathcal{C}_t(t) = \begin{bmatrix} 0 & -N_t\partial^2 \\ 0 & 0 \end{bmatrix}.$$

Thus, if $X = \begin{bmatrix} v \\ w \end{bmatrix} \in D(\mathcal{A}_0(t))$, then (recalling the operators B_0 and Γ_0 introduced earlier in this section) we have that for all $t \in [0, T]$,

$$(2.25) \quad \begin{aligned} \|\mathcal{C}_t(t)X\|_{\mathcal{H}}^2 &= \|N_t w''\|^2 \leq c\|N w''\|^2 \\ &\leq c(\|N w''\|^2 + \|B_0 v' - b_2^2 \Gamma_0(w)q_1\|_H^2) = c\|\mathcal{A}_0(t)X\|_{\mathcal{H}}^2. \end{aligned}$$

Here, $c = \sup_{(x,t) \in Q_T} N_t^2 N^{-2}$, and the norm $\|\cdot\|_{\mathcal{H}}$ corresponds to the inner product defined before (2.13). Lemma 2.6 follows from (2.24) and (2.25). $\square$

An argument similar to that for Lemma 2.5 yields the following lemma.

LEMMA 2.7. *Assume that $u = (u_1, u_2) \in C^2[0, T] \times C^1[0, T]$. There is a function g on Q_T that satisfies the conditions (2.4) such that*

$$\mathcal{C}(\cdot)\mathcal{F}_0(\cdot) \in C([0, T], \mathcal{H}_1),$$

where $\mathcal{F}_0(t)$ is defined in (2.16).

After we apply [8] to the problem (2.23) and go back all the way to the problem (2.2), we actually obtain the following theorem.

THEOREM 2.1. *Suppose that $T > 0$ and $u = (u_1, u_2) \in C^3[0, T] \times C^2[0, T]$. Assume that $(\varphi_0, \psi_0) \in H^2(0, L) \times H^3(0, L)$ satisfies the boundary conditions*

$$\varphi_0(L) = \psi_0(L) = \psi_0''(L) = 0$$

and the compatibility conditions

$$(2.26) \quad \begin{cases} 2b_1 u_1(0) = \varphi_0(0) - b_1^2 \psi_0'(0), \\ 2b_2 u_2(0) = 2\kappa b_1^{-1} u_{1t}(0) + \varphi_0'(0) - \kappa b_1^{-2} N(0, 0) \psi_0''(0) + b_2^2 \psi_0(0). \end{cases}$$

Then the problem (2.2) admits a unique solution (φ, ψ) satisfying

$$\varphi \in \bigcap_{i=0}^2 C^i([0, T]; H^{2-i}(0, L)), \quad \psi \in \bigcap_{i=0}^3 C^i([0, T]; H^{3-i}(0, L)).$$

Remark 2.1. For an integer $k \geq 2$ the method for Theorem 2.1 can be used to find solutions to the problem (2.2) such that

$$\varphi \in \bigcap_{i=0}^k C^i([0, T]; H^{k-i}(0, L)), \quad \psi \in \bigcap_{i=0}^{k+1} C^i([0, T]; H^{k-i}(0, L)),$$

where, of course, more boundary conditions and more compatibility conditions for the initial data and the input function are needed.

3. Quasi-linear equations. In this section we use Theorem 2.1 and a fixed-point argument to establish the existence of solutions of the quasi-linear problem (1.11)–(1.12). To this end, we need to establish some energy estimates of the solutions to the linear problem (2.2), which is also useful when we derive the global solutions later.

Let $T > 0$ and $u = (u_1, u_2) \in C^3(0, T) \times C^2(0, T)$. Suppose that $(\varphi_0, \psi_0) \in H^2(0, L) \times H^3(0, L)$ satisfies the boundary conditions (1.15) and the compatibility conditions (1.16), and φ_0 satisfies the σ -condition (1.14). We fix constants $\xi_{\pm} \in \mathbb{R}$ with $\xi_- < 0 < \xi_+$ such that

$$(3.1) \quad \sigma_- < \xi_- < \inf_{x \in [0, L]} \varphi_0(x) \leq \sup_{x \in [0, L]} \varphi_0(x) < \xi_+ < \sigma_+,$$

where $\sigma_{\pm}$ are given by (1.9).

For $r, T > 0$, let $M(r, T, \xi_-, \xi_+)$ denote the set of all functions v defined on Q_T which satisfy

$$(3.2) \quad \begin{aligned} v &\in \bigcap_{i=0}^2 C^i([0, T]; H^{2-i}(0, L)), \\ \xi_- &\leq v(x, t) \leq \xi_+, \quad (x, t) \in Q_T, \end{aligned}$$

$$(3.3) \quad v(x, 0) = \varphi_0(x), \quad v_t(x, 0) = \sigma'(\sigma^{-1}(\varphi_0(x)))\psi_0''(x), \quad x \in [0, L],$$

$$(3.4) \quad \begin{aligned} v(L, t) &= 0, \quad t \in [0, T], \\ \operatorname{esssup}_{t \in [0, T]} \mathbf{h}(v, t) &\leq r^2, \end{aligned}$$

where

$$(3.5) \quad \mathbf{h}(v, t) = v'^2(0, t) + \sum_{|\alpha| \leq 2} \|\partial^{\alpha} v\|^2.$$

In the above formula $\|\cdot\|$ denotes, as usual, the norm in $L^2(0, L)$. It follows from the trace theorem that $M(r, T, \xi_-, \xi_+)$ is nonempty if r is sufficiently large and T small. For example, if we let

$$v(x, t) = \varphi_0(x) \cos t + \sigma'(\sigma^{-1}(\varphi_0(x)))\psi_0''(x) \sin t,$$

then $v \in M(r, T, \xi_-, \xi_+)$ for r suitably large and T small. In the present section, we always make this assumption.

For any $v \in M(r, T, \xi_-, \xi_+)$, we consider the linear problem

$$(3.6) \quad \begin{cases} \eta(v)\varphi_t - \psi'' = 0, \\ \psi_t - \kappa\psi_t'' + \varphi'' = 0, \\ \varphi(x, 0) = \varphi_0(x), \quad \psi(x, 0) = \psi_0(x), \\ \varphi(L, t) = \psi(L, t) = \psi''(L, t) = 0, \\ 2b_1u_1(t) = \varphi(0, t) - b_1^2\psi'(0, t), \\ 2b_2u_2(t) = \varphi'(0, t) - \kappa\psi_t'(0, t) + b_2^2\psi(0, t), \end{cases}$$

with the output

$$\begin{cases} 2b_1y_1(t) = \varphi(0, t) + b_1^2\psi'(0, t), \\ 2b_2y_2(t) = \varphi'(0, t) - \kappa\psi_t'(0, t) - b_2^2\psi(0, t), \end{cases}$$

where

$$(3.7) \quad \eta(z) = 1/\mathcal{N}(z), \quad z \in (\sigma_-, \sigma_+)$$

and $\mathcal{N}(\cdot)$ is defined in (1.8). Note that the assumptions (3.1) and (3.2) imply

$$0 < \inf_{\sigma^{-1}(\xi_-) \leq \xi \leq \sigma^{-1}(\xi_+)} \sigma'(\xi) \leq \mathcal{N}(v(x, t)) \leq \sup_{\sigma^{-1}(\xi_-) \leq \xi \leq \sigma^{-1}(\xi_+)} \sigma'(\xi) < \infty$$

for all $(x, t) \in Q_T$, so that there are $\eta_1 > 0$, $\eta_2 > 0$ such that

$$\eta_1 \leq \eta(v) \leq \eta_2 \quad \forall v \in M(r, T, \xi_-, \xi_+).$$

It is easy to check that for $v \in M(r, T, \xi_-, \xi_+)$ and $N = \mathcal{N}(v)$ the compatibility conditions (2.26) hold. By Theorem 2.1, for any $v \in M(r, T, \xi_-, \xi_+)$, the problem (3.6) has a unique solution (φ, ψ) with the regularity $\varphi \in \cap_{i=0}^2 C([0, T]; H^{2-i}(0, L))$ and $\psi \in \cap_{i=0}^3 C([0, T]; H^{3-i}(0, L))$. We denote by $\mathcal{K}$ the map which carries $v \in M(r, T, \xi_-, \xi_+)$ into φ , i.e.,

$$(3.8) \quad \mathcal{K}v = \varphi,$$

where (φ, ψ) is the solution of the problem (3.6). We establish an a priori estimate to show that $\mathcal{K}$ maps $M(r, T, \xi_-, \xi_+)$ into itself if r is sufficiently large and T is sufficiently small relative to r . We then equip $M(r, T, \xi_-, \xi_+)$ with the complete metric ρ defined by

$$\rho(v, \hat{v}) = \sup_{t \in [0, T]} \|v(\cdot, t) - \hat{v}(\cdot, t)\|$$

and show that $\mathcal{K}$ is a strict contraction if T is sufficiently small. Then the Banach fixed-point theorem implies that $\mathcal{K}$ has a fixed point which is obviously a solution to

the problem (1.11)–(1.12). We define, omitting to write $(\cdot, t)$ in many places,

$$\begin{aligned}\Gamma(t) &= \frac{1}{2} \varphi'^2(0, t), \\ E_{00}(t) &= \frac{1}{2} [\|\psi\|^2 + \kappa \|\psi'\|^2 + \langle \eta(v) \varphi, \varphi \rangle], \\ E_{10}(t) &= \frac{1}{2} [\|\psi'\|^2 + \kappa \|\psi''\|^2 + \langle \eta(v) \varphi', \varphi' \rangle], \\ E_{01}(t) &= \frac{1}{2} [\|\psi_t\|^2 + \kappa \|\psi'_t\|^2 + \langle \eta(v) \varphi_t, \varphi_t \rangle], \\ E_{20}(t) &= \frac{1}{2} [\|\psi''\|^2 + \kappa \|(\eta(v) \varphi_t)'\|^2 + \langle \eta(v) \varphi'', \varphi'' \rangle], \\ E_{02}(t) &= \frac{1}{2} [\|\psi_{tt}\|^2 + \kappa \|\psi'_{tt}\|^2 + \langle \eta(v) \varphi_{tt}, \varphi_{tt} \rangle].\end{aligned}$$

Moreover, we set

$$\begin{aligned}\wp &= b_1^2 + b_1^{-2} + b_2^2 + b_2^{-2}, \\ \aleph(v, t) &= 8\wp[(\kappa^2 + 1)(\eta(v(0, t)) + 1)^2 + 1] + 1, \\ (3.9) \quad \Pi(t) &= \Gamma(t) + \sum_{i=1}^2 E_{i0}(t) + \aleph(v, t) \sum_{i=0}^2 E_{0i}(t).\end{aligned}$$

Remark 3.1. It is clear that the function $\aleph(v, t)$ depends on the function $v(0, t)$. However, there are constants $c_1(\xi_-, \xi_+) > 0$, $c_2(\xi_-, \xi_+) > 0$, which depend only on ξ_-, ξ_+ , such that for any function v satisfying the condition (3.2),

$$c_1(\xi_-, \xi_+) \leq \aleph(v, t) \leq c_2(\xi_-, \xi_+).$$

Similar estimates hold for $\Pi(t)$. Further estimates on $\aleph$ will be given in (3.27)–(3.29).

THEOREM 3.1. *Let ξ_-, ξ_+ be such that*

$$(3.10) \quad \sigma_- < \xi_- < 0 < \xi_+ < \sigma_+,$$

where $\sigma_{\pm}$ are given by (1.9). Let $v \in \cap_{i=0}^2 C^i([0, T]; H^{2-i}(0, L))$ satisfy

$$(3.11) \quad \xi_- \leq v(x, t) \leq \xi_+, \quad (x, t) \in Q_T,$$

and the boundary condition

$$v(L, t) = 0, \quad 0 \leq t \leq T.$$

Suppose that (φ, ψ) is the solution of the problem (3.6) on $Q_T = (0, L) \times (0, T)$.

Then there is a number $C(\xi_-, \xi_+) > 0$, which may depend on the constants ξ_-, ξ_+ but is independent of the initial data (φ_0, ψ_0) , such that for $0 \leq t \leq T$

$$(3.12) \quad \frac{d\Pi(t)}{dt} \leq C(\xi_-, \xi_+) |u|_2^2(t) - |y|_2^2(t) + C(\xi_-, \xi_+) \left[\mathbf{h}^{\frac{1}{2}}(v, t) + \mathbf{h}(v, t) \right] \Pi(t),$$

where $|u|_2(t)$ is defined by (1.19) and $\mathbf{h}(v, t)$ is defined by (3.5).

Remark 3.2. Here we use a special combination of $E_{ij}(t)$, as in the formula (3.9), to control the boundary terms to have the term $-|y|_2^2$ on the right-hand side of the inequality (3.12). We need the estimate (3.12) for the existence of local smooth solutions of the quasi-linear system, and it also plays a key role in the proof of the existence of global smooth solutions.

The proof of Theorem 3.1 is divided into several lemmas. We introduce the notation ∇ for the gradient of a function φ defined on Q_T , so that $\nabla\varphi = [\varphi' \ \varphi_t]$. We denote

$$|\nabla\varphi(\cdot, t)|_\infty = \sup_{x \in [0, L]} \{|\varphi'(x, t)|, |\varphi_t(x, t)|\}.$$

LEMMA 3.1. *Let $v \in \cap_{i=0}^2 C^i([0, T]; H^{2-i}(0, L))$ satisfy the condition (3.11). Suppose that (φ, ψ) is the solution to the problem (3.6). Then*

$$(3.13) \quad |\nabla\varphi(\cdot, t)|_\infty^2 \leq 2(L+1) \cdot \mathbf{h}(\varphi, t), \quad t \in [0, T],$$

$$(3.14) \quad |\nabla v(\cdot, t)|_\infty^2 \leq 2(L+1)L \cdot \mathbf{h}(v, t), \quad t \in [0, T].$$

This follows from the boundary conditions. By the boundary structure of the system (3.6), a simple computation shows that the following lemma holds.

LEMMA 3.2. *With the notation from the previous lemma,*

$$(3.15) \quad \begin{cases} b_2\psi(0) = u_2 - y_2, \\ b_1\psi'(0) = y_1 - u_1, \\ \varphi(0) = b_1(y_1 + u_1), \\ \varphi'(0) - \kappa\psi'_t(0) = b_2(y_2 + u_2), \\ \varphi'(0) = \kappa b_1^{-1}(y_{1t} - u_{1t}) + b_2(u_2 + y_2). \end{cases}$$

In the formulas (3.15) we have suppressed the variable t for the sake of simplicity; for example, $\psi(0) = \psi(0, t)$, $\varphi(0) = \varphi(0, t)$, and so on. We shall always make this omission if there is no danger of confusion.

Denote

$$(3.16) \quad \check{c}(\xi_-, \xi_+) = \inf_{\xi_- \leq x \leq \xi_+} \eta(x), \quad \hat{c}(\xi_-, \xi_+) = \sup_{\xi_- \leq x \leq \xi_+} \eta(x),$$

$$(3.17) \quad c_1(\xi_-, \xi_+) = \sup_{\xi_- \leq x \leq \xi_+} |\eta'(x)|, \quad c_2(\xi_-, \xi_+) = \sup_{\xi_- \leq x \leq \xi_+} |\eta''(x)|.$$

Since $[\xi_-, \xi_+] \subset (\sigma_-, \sigma_+)$, the formulas (1.8), (1.9), (1.10), and (3.7) imply that

$$0 < \check{c}(\xi_-, \xi_+) \leq \hat{c}(\xi_-, \xi_+) < \infty, \quad c_1(\xi_-, \xi_+) < \infty, \quad c_2(\xi_-, \xi_+) < \infty.$$

LEMMA 3.3. *Let the assumptions in Theorem 3.1 hold. Then there exists a number $C(\xi_-, \xi_+) > 0$, which may depend on ξ_- and ξ_+ , such that, for $0 \leq t \leq T$,*

$$\frac{dE_{00}(t)}{dt} \leq |u(t)|^2 - |y(t)|^2 + C(\xi_-, \xi_+) \mathbf{h}^{\frac{1}{2}}(v, t) E_{00}(t).$$

Proof. Using the systems (3.6), the formulas (3.15), (3.16), and (3.17), and integration by parts, we find (suppressing the notation for the dependence on t)

$$\begin{aligned} \frac{dE_{00}}{dt} &= \langle \psi, \psi_t \rangle + \kappa \langle \psi', \psi'_t \rangle + \langle \varphi, \psi'' \rangle + \frac{1}{2} \langle \eta'(v) v_t \varphi, \varphi \rangle \\ &\leq (u_2 + y_2) b_2 \psi(0) - (u_1 + y_1) b_1 \psi'(0) + c_1(\xi_-, \xi_+) |\nabla v|_\infty \|\varphi\|^2 \\ &= |u|^2 - |y|^2 + c_1(\xi_-, \xi_+) |\nabla v|_\infty \|\varphi\|^2 \\ &\leq |u|^2 - |y|^2 + 2\check{c}^{-1}(\xi_-, \xi_+) c_1(\xi_-, \xi_+) [2(L+1)]^{1/2} \mathbf{h}^{\frac{1}{2}}(v) E_{00}, \end{aligned}$$

where the inequality (3.14) was used. $\square$

LEMMA 3.4. *Let the assumptions in Theorem 3.1 hold. Then there exists a number $C(\xi_-, \xi_+) > 0$, which may depend on ξ_-, ξ_+ , such that, for $0 \leq t \leq T$,*

$$(3.18) \quad \frac{dE_{01}(t)}{dt} \leq |u_t(t)|^2 - |y_t(t)|^2 + C(\xi_-, \xi_+) \mathbf{h}^{\frac{1}{2}}(v, t) E_{01}(t).$$

Proof. By differentiating the first and second equations in (3.6) with respect to the time t , we obtain

$$\begin{cases} \eta(v) \varphi_{tt} = \psi''_t - \eta'(v) v_t \varphi_t, \\ \psi_{tt} - \kappa \psi''_{tt} + \varphi''_t = 0. \end{cases}$$

This together with (3.15) yields

$$\begin{aligned} \frac{dE_{01}}{dt} &= \langle \psi_t, \psi_{tt} \rangle + \kappa \langle \psi'_t, \psi'_{tt} \rangle + \langle \eta(v) \varphi_t, \varphi_{tt} \rangle + \frac{1}{2} \langle \eta'(v) v_t \varphi_t, \varphi_t \rangle \\ &= \kappa \psi_t \psi'_{tt}|_0^L + \varphi_t \psi'_t|_0^L - \varphi'_t \psi_t|_0^L - \frac{1}{2} \langle \eta'(v) v_t \varphi_t, \varphi_t \rangle \\ &= |u_t|^2 - |y_t|^2 - \frac{1}{2} \langle \eta'(v) v_t \varphi_t, \varphi_t \rangle. \end{aligned}$$

The inequality (3.18) follows from here. $\square$

LEMMA 3.5. *Let the assumptions of Theorem 3.1 hold. Then there exists a number $C(\xi_-, \xi_+) > 0$, which depends on ξ_-, ξ_+ , such that, for $0 \leq t \leq T$,*

$$\frac{dE_{02}(t)}{dt} \leq |u_{tt}(t)|^2 - |y_{tt}(t)|^2 + C(\xi_-, \xi_+) \left[\mathbf{h}^{\frac{1}{2}}(v, t) + \mathbf{h}(v, t) \right] \mathbf{h}(\varphi, t).$$

Proof. After we differentiate the first and second equations in (3.6) twice with respect to t , we obtain (suppressing the dependence on t in the notation)

$$(3.19) \quad \begin{cases} \eta(v) \varphi_{ttt} = \psi''_{tt} - 2\eta'(v) v_t \varphi_{tt} - \eta''(v) v_t^2 \varphi_t - \eta'(v) v_{tt} \varphi_t - \eta''(v) v_t^2 \varphi_t, \\ \psi_{ttt} - \kappa \psi''_{ttt} + \varphi''_{tt} = 0. \end{cases}$$

It follows from (3.19), (3.13), and (3.14) that

$$(3.20) \quad \left\langle \eta(v) \varphi_{ttt}, \varphi_{tt} \right\rangle = \left\langle \psi''_{tt}, \varphi_{tt} \right\rangle + I_1,$$

where

$$\begin{aligned} I_1 &= \left\langle -2\eta'(v) v_t \varphi_{tt} - \eta''(v) v_{tt} \varphi_t - \eta''(v) v_t^2 \varphi_t, \varphi_{tt} \right\rangle \\ &\leq C(\xi_-, \xi_+) \left[|\nabla v|_\infty \|\varphi_{tt}\|^2 + \|v_{tt}\| |\nabla \varphi|_\infty \|\varphi_{tt}\| + |\nabla v|_\infty^2 \|\varphi_t\| \|\varphi_{tt}\| \right] \\ &\leq C(\xi_-, \xi_+) \left[\mathbf{h}^{\frac{1}{2}}(v) + \mathbf{h}(v) \right] \mathbf{h}(\varphi). \end{aligned}$$

Using this and (3.19), (3.20), (3.15), we find

$$\begin{aligned}
\frac{dE_{02}}{dt} &= \langle \psi_{ttt}, \psi_{tt} \rangle + \kappa \langle \psi'_{ttt}, \psi'_{tt} \rangle + \langle \eta(v) \varphi_{ttt}, \varphi_{tt} \rangle + \frac{1}{2} \langle \eta'(v) v_t \varphi_{tt}, \varphi_{tt} \rangle \\
&\leq \kappa \psi_{tt} \psi'_{ttt}|_0^L + \varphi_{tt} \psi'_{tt}|_0^L - \varphi'_{tt} \psi_{tt}|_0^L + I_1 + C(\xi_-, \xi_+) |\nabla v|_\infty \mathbf{h}(\varphi) \\
&= [\varphi'(0) - \kappa \psi'_t(0)]_{tt} \psi_{tt}(0) - \varphi_{tt}(0) \psi'_{tt}(0) + I_1 + C(\xi_-, \xi_+) |\nabla v|_\infty \mathbf{h}(\varphi) \\
&= (u_{2tt} + y_{2tt}) b_2 \psi_{tt}(0) - (u_{1tt} + y_{1tt}) b_1 \psi'_{tt}(0) + I_1 + C(\xi_-, \xi_+) |\nabla v|_\infty \mathbf{h}(\varphi) \\
&\leq |u_{tt}|^2 - |y_{tt}|^2 + C(\xi_-, \xi_+) \left[\mathbf{h}^{\frac{1}{2}}(v) + \mathbf{h}(v) \right] \mathbf{h}(\varphi). \quad \square
\end{aligned}$$

For $u = (u_1, u_2) \in H^1(0, T)^2$, we define $|u|_1(t)$ by

$$|u|_1^2(t) = |u(t)|^2 + |u_t(t)|^2.$$

This is similar to the notation $|u|_2(t)$ introduced in (1.19).

LEMMA 3.6. *There exists a number $C(\xi_-, \xi_+) > 0$, which depends only on ξ_- , ξ_+ , such that, for $0 \leq t \leq T$,*

$$\frac{dE_{10}(t)}{dt} \leq [2\eta(v(0, t))(\kappa^2 + 1) + 1] \wp [|u|_1^2(t) + |y|_1^2(t)] + C(\xi_-, \xi_+) \mathbf{h}^{\frac{1}{2}}(v, t) E_{10}(t).$$

Proof. From the boundary conditions and the second equation of the system (3.6), it follows that

$$(3.21) \quad \varphi''(L) = 0, \quad \varphi''(0) - \kappa \psi_t''(0) = -\psi_t(0).$$

Using the first equation of the system (3.6), we obtain

$$(3.22) \quad \eta(v) \varphi'_t = [\eta(v) \varphi_t]' - \eta'(v) v' \varphi_t = \psi^{(3)} - \eta'(v) \eta^{-1}(v) v' \psi''.$$

Using (3.22), the second equation in (3.6), (3.21), and the first equation in (3.6),

$$\begin{aligned}
\frac{dE_{10}(t)}{dt} &= \langle \psi'_t, \psi' \rangle + \kappa \langle \psi''_t, \psi'' \rangle + \langle \eta(v) \varphi'_t, \varphi' \rangle + \frac{1}{2} \langle \eta'(v) v_t \varphi', \varphi' \rangle \\
&= \kappa \psi' \psi''_t|_0^L + \varphi' \psi''|_0^L - \varphi'' \psi'|_0^L + \left\langle \psi'_t - \kappa \psi_t^{(3)} + \varphi^{(3)}, \psi' \right\rangle \\
&\quad - \langle \eta'(v) \eta^{-1}(v) v' \psi'', \varphi' \rangle + \frac{1}{2} \langle \eta'(v) v_t \varphi', \varphi' \rangle \\
&\leq -\psi_t(0) \psi'(0) - \eta(v(0)) \varphi_t(0) \varphi'(0) + C(\xi_-, \xi_+) |\nabla v|_\infty E_{10}(t) \\
&\leq \frac{1}{2} [\psi_t^2(0) + \psi'^2(0)] + \frac{1}{2} \eta(v(0)) [\varphi_t^2(0) + \varphi'^2(0)] + C(\xi_-, \xi_+) |\nabla v|_\infty E_{10}(t).
\end{aligned}$$

Moreover, a simple computation using (3.15) shows that

$$\begin{aligned}
&\frac{1}{2} [\psi_t^2(0) + \psi'^2(0)] + \frac{1}{2} \eta(v(0)) [\varphi_t^2(0) + \varphi'^2(0)] \\
&\leq [2\eta(v(0))(\kappa^2 + 1) + 1] \wp (|u|^2 + |u_t|^2 + |y|^2 + |y_t|^2). \quad \square
\end{aligned}$$

LEMMA 3.7. *There exists a number $C(\xi_-, \xi_+) > 0$, which depends only on ξ_- , ξ_+ such that, for $0 \leq t \leq T$,*

$$\begin{aligned}
\frac{dE_{20}(t)}{dt} &\leq \{4\wp [(2\kappa^2 + 1)\eta^2(v(0, t)) + 1] + \eta(v(0, t))\} [|u|_2^2(t) + |y|_2^2(t)] \\
&\quad + C(\xi_-, \xi_+) [\mathbf{h}(v, t) + \mathbf{h}^{\frac{1}{2}}(v, t)] \mathbf{h}(\varphi, t).
\end{aligned}$$

Proof. Differentiating the first equation of (3.6) twice with respect to x gives

$$\eta(v)\varphi_t'' = \psi^{(4)} - 2\eta'(v)v'\varphi_t' - \eta'(v)v''\varphi_t - \eta''(v)v'^2\varphi_t.$$

This together with Lemma 3.1 then yields

$$\langle \eta(v)\varphi_t'', \varphi'' \rangle = \langle \psi^{(4)}, \varphi'' \rangle - I_0,$$

where

$$(3.23) \quad \begin{aligned} I_0 &= \langle 2\eta'(v)v'\varphi_t' + \eta'(v)v''\varphi_t + \eta''(v)v'^2\varphi_t, \varphi'' \rangle \\ &\leq C(\xi_-, \xi_+) \left[\mathbf{h}^{\frac{1}{2}}(v) + \mathbf{h}(v) \right] \mathbf{h}(\varphi). \end{aligned}$$

Next, noting that $(\eta(v)\varphi_t)'' = \psi^{(3)}$, we obtain via integration by parts and the estimate (3.23) that

$$(3.24) \quad \begin{aligned} \frac{dE_{20}}{dt} &= \langle \psi'', \psi_t'' \rangle + \kappa \langle \psi^{(3)}, \psi_t^{(3)} \rangle + \langle \eta(v)\varphi_t'', \varphi_t'' \rangle + \frac{1}{2} \langle \eta'(v)v_t\varphi'', \varphi'' \rangle \\ &\leq \kappa \psi'' \psi_t^{(3)}|_0^L + \varphi'' \psi^{(3)}|_0^L - \varphi^{(3)} \psi''|_0^L + \langle \psi'', \psi_t'' - \kappa \psi_t^{(4)} + \varphi^{(4)} \rangle - I_0 \\ &= \left[\varphi^{(3)}(0) - \kappa \psi_t^{(3)}(0) \right] \psi''(0) - \varphi''(0) \psi^{(3)}(0) - I_0 \\ &= -\psi_t'(0) \psi''(0) - \varphi''(0) \psi^{(3)}(0) - I_0. \end{aligned}$$

From the formulas in (3.15) we see that

$$(3.25) \quad \begin{aligned} -\psi_t'(0) \psi''(0) &= -\psi_t'(0) \eta(v(0)) \varphi_t(0) = -\eta(v(0)) b_1 \psi_t'(0) (y_{1t} + u_{1t}) \\ &= \eta(v(0)) (u_{1t}^2 - y_{1t}^2). \end{aligned}$$

Let us estimate the term $-\varphi''(0) \psi^{(3)}(0)$. From the equations in (3.6) and the formulas in (3.15) we have

$$\begin{aligned} \varphi''(0) &= \kappa \psi_t''(0) - \psi_t(0) = \kappa(\eta(v)\varphi_t)_t(0) - \psi_t(0) \\ &= \kappa \eta'(v(0)) v_t(0) \varphi_t(0) + \kappa b_1 \eta(v(0)) (u_{1tt} + y_{1tt}) - b_2^{-1} (u_{2t} - y_{2t}). \end{aligned}$$

It follows from here that

$$(3.26) \quad \varphi''^2(0) \leq C(\xi_-, \xi_+) |\nabla \varphi|_\infty^2 |\nabla v|_\infty^2 + 8[\kappa^2 \eta^2(v(0)) + 1]_\wp [u_{2t}^2 + y_{2t}^2 + u_{1tt}^2 + y_{1tt}^2].$$

In addition, a similar computation gives the estimate

$$[\psi^{(3)}(0)]^2 \leq C(\xi_-, \xi_+) |\nabla \varphi|_\infty^2 |\nabla v|_\infty^2 + 8\eta^2(v(0))(\kappa^2 + 1)_\wp [u_{1tt}^2 + y_{1tt}^2 + u_{2t}^2 + y_{2t}^2].$$

Combining this with (3.26), (3.25), (3.24), and (3.23) completes the proof. $\square$

A similar computation proves the following lemma.

LEMMA 3.8.

$$\frac{d\Gamma(t)}{dt} \leq 4(\kappa^2 + 1)_\wp [|u|_2^2 + |y|_2^2].$$

Proof of Theorem 3.1. Under the assumption (3.11), there are $c(\xi_-, \xi_+) > 0$, $C(\xi_-, \xi_+) > 0$, which may depend on ξ_-, ξ_+ but are independent of the initial data and the input, such that, if v satisfies the condition (3.2), then

$$(3.27) \quad 1 \leq \aleph(v, t) \leq C(\xi_-, \xi_+),$$

$$(3.28) \quad \left| \frac{d\aleph(v, t)}{dt} \right| \leq C(\xi_-, \xi_+) \mathbf{h}^{\frac{1}{2}}(v, t),$$

$$(3.29) \quad \mathbf{h}(\varphi, t) \leq C(\xi_-, \xi_+) \Pi(t)$$

for all $t \in [0, T]$. By summing up the differential inequalities from Lemmas 3.3–3.8 and using (3.27)–(3.29), we compute that for $t \in [0, T]$,

$$\begin{aligned} \frac{d\Pi(t)}{dt} &= \frac{d\Gamma(t)}{dt} + \sum_{i=1}^2 \frac{dE_{i0}(t)}{dt} + \aleph(v, t) \sum_{i=0}^2 \frac{dE_{0i}(t)}{dt} + \frac{d\aleph(v, t)}{dt} \sum_{i=0}^2 E_{0i}(t) \\ &\leq [\aleph(v, t) - 1] [|u|_2^2(t) + |y|_2^2(t)] + \aleph(v, t) [|u|_2^2(t) - |y|_2^2(t)] \\ &\quad + C(\xi_-, \xi_+) [\mathbf{h}(v, t) + \mathbf{h}^{\frac{1}{2}}(v, t)] \Pi(t) \\ &= [2\aleph(v, t) - 1] |u|_2^2(t) - |y|_2^2(t) + C(\xi_-, \xi_+) [\mathbf{h}(v, t) + \mathbf{h}^{\frac{1}{2}}(v, t)] \Pi(t) \\ &\leq C(\xi_-, \xi_+) |u|_2^2(t) - |y|_2^2(t) + C(\xi_-, \xi_+) [\mathbf{h}(v, t) + \mathbf{h}^{\frac{1}{2}}(v, t)] \Pi(t). \quad \square \end{aligned}$$

Proof of Theorem 1.1. Since φ_0 satisfies the σ -condition (1.14), we can fix constants ξ_-, ξ_+ such that the relations (3.1) and (3.10) are true.

Recall the notation $M(r, T, \xi_-, \xi_+)$ from the beginning of this section. Fix r large and T small such that $M(r, T, \xi_-, \xi_+)$ is nonempty. For any $v \in M(r, T, \xi_-, \xi_+)$, by Theorem 2.1, the condition (3.2) implies that the problem (3.6) has a unique solution (φ, ψ) . We need to verify that $\varphi \in M(r, T, \xi_-, \xi_+)$ and the map $\mathcal{K}$, given by (3.8), is a strict contraction if r is sufficiently large and T is small.

By Theorem 3.1, there exists $C(\xi_-, \xi_+) > 0$ such that

$$\Pi(t) \leq \Pi(0) + C(\xi_-, \xi_+) \|u\|_{H^2[0, T]}^2 + C(\xi_-, \xi_+) \int_0^t [\mathbf{h}^{\frac{1}{2}}(v, \tau) + \mathbf{h}(v, \tau)] \Pi(\tau) d\tau$$

for any $t \in [0, T]$. It follows from here, (3.29), and Gronwall's inequality that

$$(3.30) \quad \mathbf{h}(\varphi, t) \leq C(\xi_-, \xi_+) \Pi(t) \leq C(\xi_-, \xi_+) [\Pi(0) + C(\xi_-, \xi_+) \|u\|_{H^2[0, T]}^2] e^{C(\xi_-, \xi_+) \int_0^t [\mathbf{h}^{\frac{1}{2}}(v, \tau) + \mathbf{h}(v, \tau)] d\tau}$$

for any $t \in [0, T]$. From the initial data (3.3) and the system (3.6), we can check that the constant $\Pi(0)$ is independent of the choice of $v \in M(r, T, \xi_-, \xi_+)$.

We now take $r_0 > r$ sufficiently large such that

$$r_0^2 \geq 2C(\xi_-, \xi_+) [\Pi(0) + C(\xi_-, \xi_+) \|u\|_{H^2[0, T]}^2].$$

It follows from (3.6), (3.4), (3.30), and the above estimate that

$$\mathbf{h}(\varphi, t) \leq \frac{r_0^2}{2} e^{C(\xi_-, \xi_+) r_0(1+r_0)t}, \quad t \in [0, T],$$

for any $v \in M(r_0, T, \xi_-, \xi_+)$. Using this, we select $T_1 \in (0, T]$ such that for $t \in (0, T_1]$ we have

$$(3.31) \quad \operatorname{esssup}_{s \in [0, t]} \mathbf{h}(\varphi, s) \leq r_0^2.$$

To find a $T_2 \in (0, T_1]$ such that $\varphi \in M(r_0, T_2, \xi_-, \xi_+)$, the only thing left to do is to verify that the condition (3.2) is satisfied by φ , since the conditions (3.3) are implied by the equations (3.6). In fact, by (3.31), for all $(x, t) \in Q_{T_1}$,

$$|\varphi(x, t) - \varphi_0(x)| = \left| \int_0^t \varphi_t(x, t) dt \right| \leq L^{1/2} \int_0^t \mathbf{h}^{\frac{1}{2}}(\varphi, s) ds \leq tL^{1/2}r_0.$$

Thus (3.2) is true for φ for small $T_2 \in (0, T_1]$, because of the assumption (3.1). With the above choice of r_0 and T_2 , $\mathcal{K}$ maps $M(r_0, t, \xi_-, \xi_+)$ into itself for every $t \in (0, T_2]$.

Now we shall find a $T_3 \in (0, T_2]$ such that $\mathcal{K}$ is contractive from $M(r_0, T_3, \xi_-, \xi_+)$ to itself. Let $v_1, v_2 \in M(r_0, T_2, \xi_-, \xi_+)$ and set $\varphi_1 = \mathcal{K}v_1$, $\varphi_2 = \mathcal{K}v_2$, $v = v_1 - v_2$, $\varphi = \varphi_1 - \varphi_2$, $\psi = \psi_1 - \psi_2$. A simple computation shows that

$$(3.32) \quad \begin{cases} \eta(v_1)\varphi_t = \psi'' - [\eta(v_1) - \eta(v_2)]\varphi_{2t}, \\ \psi_t - \kappa\psi_t'' + \varphi'' = 0, \\ \varphi(x, 0) = 0, \quad \psi(x, 0) = 0, \\ \varphi(L, t) = \psi(L, t) = \psi''(L, t) = 0, \\ \varphi(0, t) - b_1^2\psi'(0, t) = 0, \\ \varphi'(0, t) - \kappa\psi_t'(0, t) + b_2^2\psi(0, t) = 0. \end{cases}$$

Denoting

$$(3.33) \quad \hat{E}_{00}(t) = \frac{1}{2} [\|\psi\|^2 + \kappa\|\psi'\|^2 + \langle \eta(v_1)\varphi, \varphi \rangle]$$

(where the argument $(\cdot, t)$ has been omitted in several places), from (3.32) we obtain

$$(3.34) \quad \begin{aligned} \frac{d\hat{E}_{00}}{dt} &= \kappa\psi\psi_t'|_0^L + \varphi\psi'|_0^L - \varphi'\psi|_0^L + I_2 \\ &= -b_2^2\psi^2(0, t) - b_1^2\psi'^2(0, t) + I_2 \leq I_2, \end{aligned}$$

where

$$I_2 = \langle [\eta(v_2) - \eta(v_1)]\varphi_{2t}, \varphi \rangle + \frac{1}{2} \langle \eta'(v_1)v_{1t}\varphi, \varphi \rangle.$$

It follows from here and Lemma 3.1 that there exists a number $C(\xi_-, \xi_+) > 0$, which may depend on ξ_-, ξ_+ , such that

$$(3.35) \quad \begin{aligned} |I_2| &\leq C(\xi_-, \xi_+) [|\nabla\varphi_2|_\infty\|\varphi\| \|v_1 - v_2\| + |\nabla v_1|_\infty\|\varphi\|^2] \\ &\leq C(\xi_-, \xi_+) \left[\mathbf{h}^{\frac{1}{2}}(\varphi_2)\rho(\mathcal{K}v_1, \mathcal{K}v_2)\rho(v_1, v_2) + \mathbf{h}^{\frac{1}{2}}(v_1)\|\varphi\|^2 \right] \\ &\leq C(\xi_-, \xi_+) \left[r_0\rho(\mathcal{K}v_1, \mathcal{K}v_2)\rho(v_1, v_2) + \mathbf{h}^{\frac{1}{2}}(v_1)\|\varphi\|^2 \right], \end{aligned}$$

since $\varphi_2 = \mathcal{K}v_2 \in M(r_0, T_2, \xi_-, \xi_+)$. By virtue of (3.32), (3.33), (3.34), and (3.35),

for $T_3 \in (0, T_2]$ and for $t \in (0, T_3]$,

$$\begin{aligned} \|\varphi(\cdot, t)\|^2 &\leq C(\xi_-, \xi_+) \hat{E}_{00}(t) \leq C(\xi_-, \xi_+) \int_0^t I_2(s) ds \\ &\leq T_3 r_0 C(\xi_-, \xi_+) \rho(\mathcal{K}v_1, \mathcal{K}v_2) \rho(v_1, v_2) + C(\xi_-, \xi_+) \int_0^t \mathbf{h}^{\frac{1}{2}}(v_1, s) \|\varphi(\cdot, s)\|^2 ds. \end{aligned}$$

Applying Gronwall's inequality to the last estimate yields

$$\|\varphi(\cdot, t)\|^2 \leq T_3 r_0 C(\xi_-, \xi_+) \rho(\mathcal{K}v_1, \mathcal{K}v_2) \rho(v_1, v_2) e^{C(\xi_-, \xi_+) r_0 T_3}$$

for all $t \in (0, T_3]$, where $T_3 \in (0, T_2]$. Since $\varphi = \mathcal{K}v_1 - \mathcal{K}v_2$,

$$\rho(\mathcal{K}v_1, \mathcal{K}v_2) \leq T_3 r_0 C(\xi_-, \xi_+) \rho(v_1, v_2) e^{C(\xi_-, \xi_+) r_0 T_3}, \quad T_3 \in (0, T_2].$$

The last inequality shows that $\mathcal{K}$ is strictly contractive if T_3 is sufficiently small. By the Banach fixed-point theorem, for $T_3 > 0$ sufficiently small $\mathcal{K}$ has a unique fixed point $\varphi \in M(r_0, T_3, \xi_-, \xi_+)$. Then (φ, ψ) is the desired solution of the system (1.11)–(1.12). $\square$

Let (φ, ψ) be a solution of (1.11)–(1.12) on Q_T such that (1.17) holds. By the argument above, (φ, ψ) is the solution of the problem (3.6) on Q_T where $v = \varphi$. Thus, by the inequality (3.29), Theorem 3.1 can be rewritten as follows.

THEOREM 3.1*. *Let $T > 0$ and $\sigma_- < \xi_- < 0 < \xi_+ < \sigma_+$. There exists $C(\xi_-, \xi_+) > 0$, which depends on ξ_-, ξ_+ such that, if the solution (φ, ψ) of the problem (1.11)–(1.12) on Q_T satisfies $\xi_- \leq \varphi \leq \xi_+$, then for $0 \leq t \leq T$,*

$$(3.36) \quad \frac{d\Pi(t)}{dt} + |y|_2^2(t) \leq C(\xi_-, \xi_+) |u|_2^2(t) + C(\xi_-, \xi_+)^2 \left[\Pi^2(t) + \Pi^{3/2}(t) \right],$$

where $\Pi(t)$ is given by the formula (3.9) with $\aleph(v, t) = \aleph(\varphi, t)$.

4. Estimates of the boundary trace. Let $T > 0$ and $u = (u_1, u_2) \in H^2(0, T) \times H^2(0, T)$. Suppose that $(\varphi_0, \psi_0) \in H^2(0, L) \times H^3(0, L)$ satisfies the boundary conditions (1.15) and the compatibility conditions (1.16), where φ_0 satisfies the σ -condition (1.14). We assume that (φ, ψ) solves the problem (1.11)–(1.12) on $Q_T = (0, L) \times (0, T)$ such that the regularity (1.17) holds. Let $\xi_-, \xi_+ \in \mathbb{R}$ be such that

$$\sigma_- < \xi_- < 0 < \xi_+ < \sigma_+.$$

We define

$$(4.1) \quad T_{\xi_-, \xi_+} = L\kappa^{\frac{1}{2}} \sup_{\xi_- \leq z \leq \xi_+} \eta^{\frac{1}{2}}(z),$$

where (as usual) $\eta = 1/\mathcal{N}$ and $\mathcal{N}$ is defined in (1.8).

The purpose of this section is to prove the following theorem.

THEOREM 4.1. *Suppose that $\varsigma > 0$, $T > 3T_{\xi_-, \xi_+}$ and (φ, ψ) is a solution of the problem (1.11)–(1.12) on Q_T . Then there exists a number $C(\xi_-, \xi_+, \varsigma) > 0$, which depends on ξ_-, ξ_+ and ς such that, if (φ, ψ) satisfies*

$$\xi_- \leq \varphi \leq \xi_+, \quad \check{\mathcal{E}}_T \leq \varsigma,$$

where

$$(4.2) \quad \check{\mathcal{E}}_T = \sup_{0 \leq t \leq T} \mathcal{E}(t)$$

and $\mathcal{E}(t)$ is defined by (1.18), then for $0 \leq s \leq t \leq T$ with $t - s \geq 3T_{\xi_-, \xi_+}$,

$$(4.3) \quad \int_s^t \mathcal{E}(\tau) d\tau \leq C(\xi_-, \xi_+, \varsigma) \left(\|u\|_{H^2(s,t)}^2 + \|y\|_{H^2(s,t)}^2 + \int_s^t [\mathcal{E}^2(\tau) + \mathcal{E}^{\frac{3}{2}}(\tau)] d\tau \right).$$

Remark 4.1. In fact, for any $\varepsilon > 0$, we can prove that the inequality (4.3) holds for $t - s \geq 2T_{\xi_-, \xi_+} + \varepsilon$, where the constant $C(\xi_-, \xi_+, \varsigma)$ may also depend on ε . In general, the conclusion of Theorem 4.1 is not true when $t - s < 2T_{\xi_-, \xi_+}$.

The proof of Theorem 4.1 is based on several lemmas below. Set

$$(4.4) \quad \vartheta = \begin{bmatrix} \varphi \\ \psi' \end{bmatrix}, \quad A(\varphi) = \begin{bmatrix} 0 & -\kappa \\ -\eta(\varphi) & 0 \end{bmatrix},$$

where $\eta(z) = 1/\mathcal{N}(z) = 1/\sigma'(\sigma^{-1}(z))$. We diagonalize $A(\varphi)$ as

$$SA(\varphi)S^{-1} = \begin{bmatrix} \kappa^{\frac{1}{2}}\eta^{\frac{1}{2}}(\varphi) & 0 \\ 0 & -\kappa^{\frac{1}{2}}\eta^{\frac{1}{2}}(\varphi) \end{bmatrix},$$

where

$$(4.5) \quad S = \frac{1}{2} \begin{bmatrix} 1 & -\kappa^{\frac{1}{2}}\eta^{-\frac{1}{2}}(\varphi) \\ 1 & \kappa^{\frac{1}{2}}\eta^{-\frac{1}{2}}(\varphi) \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 1 \\ -\kappa^{-\frac{1}{2}}\eta^{\frac{1}{2}}(\varphi) & \kappa^{-\frac{1}{2}}\eta^{\frac{1}{2}}(\varphi) \end{bmatrix}.$$

LEMMA 4.1. Let $\xi_- \leq \varphi \leq \xi_+$ hold on Q_T . Let $\vartheta = \begin{bmatrix} \varphi \\ \psi' \end{bmatrix}$, where (φ, ψ) is the solution of the problem (1.11)–(1.12). Then

$$(4.6) \quad A(\varphi)\vartheta_t + \vartheta' = \begin{bmatrix} \Lambda\varphi - \kappa b_2[u_2(t) + y_2(t)]q_1' \\ 0 \end{bmatrix}, \quad (x, t) \in Q_T,$$

where

$$(4.7) \quad (\Lambda\varphi)(x) = a^{-1/2} \langle \varphi, q_0' \rangle \sinh a^{\frac{1}{2}}x - a \int_0^x \varphi(s) \cosh a^{\frac{1}{2}}(x-s) ds.$$

Proof. From the second equation in (1.11), a computation as in (2.10) yields

$$\psi_t = B_0\varphi' - b_2^2\psi(0, t)q_1 + 2b_2u_2(t)q_1,$$

where B_0 is defined in Lemma 2.1 and q_1 by (2.11). We then employ (2.9), (2.8), and $b_2\psi(0, t) = u_2(t) - y_2(t)$ (see (3.15) with $\psi = w$) to obtain

$$\begin{aligned} \kappa\psi_t' &= \kappa \frac{\partial}{\partial x} B_0\varphi' - \kappa b_2^2\psi(0, t)q_1' + 2\kappa b_2u_2(t)q_1' \\ &= \varphi' + a^{-\frac{1}{2}} \langle \varphi', q_0 \rangle \sinh a^{\frac{1}{2}}x + a^{\frac{1}{2}} \int_0^x \varphi'(s) \sinh a^{\frac{1}{2}}(x-s) ds \\ &\quad - \kappa b_2^2\psi(0, t)q_1' + 2\kappa b_2u_2(t)q_1' \\ (4.8) \quad &= \varphi' - \Lambda\varphi + \kappa b_2[u_2(t) + y_2(t)]q_1'. \end{aligned}$$

The formula (4.6) follows from (4.8) and the first equation in (1.11). $\square$

Remark 4.2. Assume for a moment that $b_2 = 0$. Using (4.8) we observe that, if we make the change of unknowns

$$\mu = \varphi, \quad \nu = \psi',$$

then the problem (1.11)–(1.12) becomes

$$(4.9) \quad \begin{cases} \mu_t = \eta^{-1}(\mu)\nu', \\ \kappa\nu_t = \mu' - \Lambda(\mu), \\ \mu(x, 0) = \varphi_0(x), \quad \nu(x, 0) = \psi_0(x), \\ \mu(L, t) = 0, \end{cases}$$

with the input-output at $x = 0$:

$$(4.10) \quad \begin{cases} 2b_1 u_1(t) = \mu(0, t) - b_1^2 \nu(0, t), \\ 2b_1 y_1(t) = \mu(0, t) + b_1^2 \nu(0, t). \end{cases}$$

It is clear that the problem (4.9)–(4.10) represents a quasi-linear string with the input-output structure at the boundary. The existence of global solutions of the above problem has been studied in [18].

If $b_2 \neq 0$, then the term $\kappa[\mu'(0, t) - \kappa\nu'_t(0, t)]q'_1$ appears on the right-hand side of the second equation in (4.9), and this makes things quite different.

For $(x, t) \in Q_T$ let $\tau_i(s) = \tau_i(s; x, t)$ be the solutions of

$$(4.11) \quad \frac{d\tau_i}{ds} = \lambda_i(s, \tau_i), \quad \tau_i(x; x, t) = t, \quad i = 1, 2,$$

where

$$\lambda_1(x, t) = \kappa^{\frac{1}{2}}\eta^{\frac{1}{2}}(\varphi(x, t)), \quad \lambda_2(x, t) = -\kappa^{\frac{1}{2}}\eta^{\frac{1}{2}}(\varphi(x, t)).$$

Then $(s, \tau_i(s; x, t))$ are the forward and backward characteristics through (x, t) , respectively, for $i = 1, 2$. An argument similar to that in [1] implies the following lemma.

LEMMA 4.2. For $i = 1, 2$,

$$\tau'_i(s; x, t) = -\lambda_i(x, t)\tau_{it}(s; x, t).$$

Suppose that $\xi_- \leq \varphi \leq \xi_+$ holds on Q_T . There exist numbers $C(\xi_-, \xi_+) > 0$ and $c(\xi_-, \xi_+) > 0$, which depend only on ξ_-, ξ_+ such that for $0 \leq s \leq x \leq L$, $0 \leq t \leq T$,

$$(4.12) \quad e^{-c(\xi_-, \xi_+)\tilde{\mathcal{E}}_T^{\frac{1}{2}}} \leq \tau_{it}(s; x, t) \leq e^{c(\xi_-, \xi_+)\tilde{\mathcal{E}}_T^{\frac{1}{2}}}$$

and

$$C^{-1}(\xi_-, \xi_+)e^{-c(\xi_-, \xi_+)\tilde{\mathcal{E}}_T^{\frac{1}{2}}} \leq |\tau'_i(s; x, t)| \leq C(\xi_-, \xi_+)e^{c(\xi_-, \xi_+)\tilde{\mathcal{E}}_T^{\frac{1}{2}}},$$

where $\tilde{\mathcal{E}}$ is defined in (4.2).

LEMMA 4.3. Assume that $\xi_- \leq \varphi \leq \xi_+$ holds on Q_T and let T_{ξ_-, ξ_+} be given by (4.1). Let $\omega \in H^1(Q_T) \times H^1(Q_T)$ such that

$$(4.13) \quad A(\varphi)\omega_t + \omega' = \Psi(\varphi), \quad (x, t) \in Q_T,$$

where $A(\varphi)$ is defined in (4.4) and $\Psi(\varphi)$ satisfies the estimate

$$(4.14) \quad |\Psi(\varphi)| \leq C(\xi_-, \xi_+) [|\Psi_0(\varphi)| + |\nabla\varphi|_\infty |\Psi_1(\varphi)| + |\nabla\varphi|_\infty^3 + |\nabla\varphi|_\infty^2 + |\nabla\varphi|_\infty |\omega|]$$

for $(x, t) \in Q_T$, where $\Psi_0(\varphi)$ and $\Psi_1(\varphi)$ (the components of $\Psi(\varphi)$) are functions determined by φ . Set

$$(4.15) \quad \Upsilon = S\omega,$$

where S is given in (4.5). Then there exist numbers $C(\xi_-, \xi_+) > 0$ and $c(\xi_-, \xi_+) > 0$, which depend only on ξ_-, ξ_+ , such that for $2T_{\xi_-, \xi_+} \leq t \leq T$

$$(4.16) \quad \int_0^t \|\Upsilon(\cdot, \tau)\|^2 d\tau \leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t [|\Upsilon(0, \varsigma)|^2 + |\Upsilon(L, \varsigma)|^2 + \Xi(\Upsilon)(\varsigma)] d\varsigma,$$

where $\tilde{\mathcal{E}}_T$ is given in (4.2) and

$$(4.17) \quad \Xi(\Upsilon)(t) = \|\Psi_0(\varphi)(\cdot, t)\|^2 + \|\Psi_1(\varphi)(\cdot, t)\|^2 + [|\nabla\varphi(\cdot, t)|_\infty^2 + |\nabla\varphi(\cdot, t)|_\infty] \|\Upsilon(\cdot, t)\|^2 + |\nabla\varphi(\cdot, t)|_\infty^4 + |\nabla\varphi(\cdot, t)|_\infty^3.$$

Furthermore, denoting the components of Υ by Υ_1, Υ_2 , we have for $2T_{\xi_-, \xi_+} \leq t \leq T$,

$$(4.18) \quad \int_{T_\zeta}^t \Upsilon_1^2(L, \tau) d\tau \leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t [\Upsilon_1^2(0, \varsigma) + \Xi(\Upsilon)(\varsigma)] d\varsigma,$$

$$(4.19) \quad \int_0^{T_\zeta} \Upsilon_2^2(L, \tau) d\tau \leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t [\Upsilon_2^2(0, \varsigma) + \Xi(\Upsilon)(\varsigma)] d\varsigma.$$

Proof. From the formulas (4.13) and (4.15), a simple computation yields

$$(4.20) \quad SAS^{-1}\Upsilon_t + \Upsilon' = \Theta,$$

where

$$(4.21) \quad \Theta = (SAS^{-1}S_t + S') S^{-1}\Upsilon + S\Psi(\varphi).$$

Thus we have from (4.20) that for $(x, t) \in Q_T$,

$$(4.22) \quad \kappa^{\frac{1}{2}} \eta^{\frac{1}{2}}(\varphi) \Upsilon_{1t} + \Upsilon'_1 = \Theta \cdot e_1, \quad -\kappa^{\frac{1}{2}} \eta^{\frac{1}{2}}(\varphi) \Upsilon_{2t} + \Upsilon'_2 = \Theta \cdot e_2,$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Let $(x, t) \in Q_T$ be given. Using (4.22), we integrate Υ_1^2 along the forward characteristic $(s, \tau_1(s))$ from 0 to x to obtain

$$(4.23) \quad \begin{aligned} \Upsilon_1^2(x, t) &= \Upsilon_1^2(0, \tau_1(0; x, t)) + 2 \int_0^x \Upsilon_1 \frac{d}{ds} \Upsilon_1 ds \\ &= \Upsilon_1^2(0, \tau_1(0; x, t)) + 2 \int_0^x (\Upsilon_1 \Theta \cdot e_1)(s, \tau_1(s)) ds. \end{aligned}$$

On the other hand, from (4.21) and (4.14) we can derive that for $(x, t) \in Q_T$,

$$|\Theta| \leq C(\xi_-, \xi_+) [|\Psi_0(\varphi)| + |\nabla\varphi|_\infty |\Psi_1(\varphi)| + |\nabla\varphi|_\infty^3 + |\nabla\varphi|_\infty^2 + |\nabla\varphi|_\infty |\Upsilon|].$$

By combining the above estimate with (4.23), we obtain

$$(4.24) \quad \Upsilon_1^2(x, t) \leq \Upsilon_1^2(0, \tau_1(0; x, t)) + C(\xi_-, \xi_+) \int_0^x f(s, \tau_1(s; x, t)) \, ds,$$

where

$$\begin{aligned} f(x, t) = & |\Upsilon_1(x, t)| \cdot |\Psi_0(\varphi)(x, t)| + |\Psi_1(\varphi)(x, t)| \cdot |\nabla\varphi(\cdot, t)|_\infty |\Upsilon_1(x, t)| \\ & + [|\nabla\varphi(\cdot, t)|_\infty^2 + |\nabla\varphi(\cdot, t)|_\infty] |\Upsilon(x, t)|^2 + |\nabla\varphi(\cdot, t)|_\infty^3 + |\nabla\varphi(\cdot, t)|_\infty^4. \end{aligned}$$

We now integrate both sides of (4.24) from T_{ξ_-, ξ_+} to t with respect to the time (denoted ν) and arrive at

$$(4.25) \quad \begin{aligned} \int_{T_{\xi_-, \xi_+}}^t \Upsilon_1^2(x, \nu) \, d\nu & \leq \int_{T_{\xi_-, \xi_+}}^t \Upsilon_1^2(0, \tau_1(0; x, \nu)) \, d\nu \\ & + C(\xi_-, \xi_+) \int_0^x \int_{T_{\xi_-, \xi_+}}^t f(s, \tau_1(s; x, \nu)) \, d\nu \, ds. \end{aligned}$$

Now we estimate the two terms on the right-hand side of (4.25) by a change of the integration variable. By the choice of the constant T_{ξ_-, ξ_+} , it is easy to see that

$$(4.26) \quad 0 \leq \tau_1(s; x, t) \leq t, \quad s \in [0, x], \quad x \in [0, L], \quad t \in [T_{\xi_-, \xi_+}, T].$$

For each fixed $s \in [0, x]$, we make a change of the integration variable by

$$\varsigma = \tau_1(s; x, t)$$

and use the inequalities (4.12) and (4.26) to obtain

$$\int_{T_{\xi_-, \xi_+}}^t f(s, \tau_1(s; x, \nu)) \, d\nu = \int_{\tau_1(s; x, T_{\xi_-, \xi_+})}^{\tau_1(s; x, t)} \frac{f(s, \varsigma)}{\tau_{1t}(s; x, \tau_1^{-1}(\varsigma))} \, d\varsigma \leq e^{c(\xi_-, \xi_+) \mathcal{E}_T^{\frac{1}{2}}} \int_0^t f(s, \nu) \, d\nu.$$

Similarly, we have

$$\int_{T_{\xi_-, \xi_+}}^t \Upsilon_1^2(0, \tau_1(0; x, \nu)) \, d\nu \leq e^{c(\xi_-, \xi_+) \mathcal{E}_T^{\frac{1}{2}}} \int_0^t \Upsilon_1^2(0, \nu) \, d\nu.$$

We now substitute the last two inequalities into (4.25) and then integrate both sides from 0 to L with respect to the variable x to obtain

$$(4.27) \quad \begin{aligned} \int_{T_{\xi_-, \xi_+}}^t \|\Upsilon_1(\cdot, \nu)\|^2 \, d\nu & \leq \varepsilon \int_0^t \|\Upsilon_1(\cdot, \nu)\|^2 \, d\nu \\ & + C(\xi_-, \xi_+, \varepsilon) e^{c(\xi_-, \xi_+) \mathcal{E}_T^{\frac{1}{2}}} \int_0^t [\Upsilon_1^2(0, \varsigma) + \Xi(\Upsilon)(\varsigma)] \, d\varsigma, \end{aligned}$$

where we have used the inequality $|\Upsilon_1| |\Psi_0(\varphi)| \leq \varepsilon |\Upsilon_1|^2 + |\Psi_0(\varphi)|^2 / 4\varepsilon$ for ε small.

Clearly, in order to obtain an estimate for $\int_0^t \|\Upsilon_1(\cdot, \nu)\|^2 d\nu$, we have to estimate $\int_0^{T_{\xi_-, \xi_+}} \|\Upsilon_1(\cdot, \nu)\|^2 d\nu$. To this end, we repeat the above process by using the forward characteristic $(s, \tau_1(s))$ again but this time on the segment $x \leq s \leq L$. We integrate $\Upsilon_1^2(s, \tau_1(s))$ from x to L and then we have

$$(4.28) \quad \Upsilon_1^2(x, t) \leq \Upsilon_1^2(L, \tau_1(L; x, t)) + C(\xi_-, \xi_+) \int_0^L f(s, \tau_1(s; x, t)) ds.$$

From the choice of the T_{ξ_-, ξ_+} it is easy to see that

$$t \leq \tau_1(s; x, t) \leq \tau_1(L; x, t) \leq T_{\xi_-, \xi_+} + t, \quad s \in [x, L], \quad t \in [0, T].$$

We integrate both sides of (4.28) from 0 to T_{ξ_-, ξ_+} with respect to t . Noting that

$$\begin{aligned} \int_0^{T_{\xi_-, \xi_+}} \Upsilon_1^2(L, \tau_1(L; x, t)) dt &= \int_{\tau_1(L; x, 0)}^{\tau_1(L; x, T_{\xi_-, \xi_+})} \Upsilon_1^2(L, \varsigma) \tau_{1t}^{-1} d\varsigma \\ &\leq e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^{2T_{\xi_-, \xi_+}} \Upsilon_1^2(L, t) dt, \end{aligned}$$

a computation similar to that for the inequality (4.27) yields

$$\begin{aligned} \int_0^{T_{\xi_-, \xi_+}} \|\Upsilon_1(\cdot, t)\|^2 dt &\leq \varepsilon \int_0^{2T_{\xi_-, \xi_+}} \|\Upsilon_1(\cdot, t)\|^2 dt \\ &\quad + C(\xi_-, \xi_+, \varepsilon) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^{2T_{\xi_-, \xi_+}} [\Upsilon_1^2(L, \varsigma) + \Xi(\Upsilon)(\varsigma)] d\varsigma. \end{aligned}$$

It follows from the above inequality and (4.27) that for $2T_{\xi_-, \xi_+} \leq t \leq T$,

$$\int_0^t \|\Upsilon_1(\cdot, \nu)\|^2 d\nu \leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t [\Upsilon_1^2(0, \varsigma) + \Upsilon_1^2(L, \varsigma) + \Xi(\Upsilon)(\varsigma)] d\varsigma.$$

Now, we do the same thing along the backward characteristic $(s, \tau_2(s))$ and then we obtain an inequality which is the same as the last one if Υ_1 is changed into Υ_2 . Therefore, addition of these two inequalities actually yields (4.16).

Next, we prove the inequalities (4.18) and (4.19). We go back to the inequality (4.24) and set $x = L$ in (4.24). Then we integrate both sides of (4.24) with respect to time from T_{ξ_-, ξ_+} to t to give the inequality (4.18). As to the inequality (4.19), we do the same thing on the backward characteristic $(s, \tau_2(s))$ ($0 \leq s \leq L$). Noting the inequality

$$0 \leq \tau_2(0; L, 0) \leq \tau_2(0; L, T_{\xi_-, \xi_+}) \leq 2T_{\xi_-, \xi_+} \leq t,$$

we obtain the inequality (4.19). $\square$

LEMMA 4.4. *Assume that $\xi_- \leq \varphi \leq \xi_+$ holds on Q_T . If $T > 2T_{\xi_-, \xi_+}$, then there exist numbers $C(\xi_-, \xi_+) > 0$ and $c(\xi_-, \xi_+) > 0$, which depend only on ξ_-, ξ_+ , such that for $2T_{\xi_-, \xi_+} \leq t \leq T$,*

$$(4.29) \quad \begin{aligned} \int_0^t \mathcal{E}(\tau) d\tau &\leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \left(\|u\|_{H^2(0, t)}^2 + \|y\|_{H^2(0, t)}^2 \right) \\ &\quad + C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t \left[\mathbf{h}(\Lambda\varphi, \tau) + \mathcal{E}^2(\tau) + \mathcal{E}^{\frac{3}{2}}(\tau) \right] d\tau, \end{aligned}$$

where $\Lambda\varphi$ and $\mathbf{h}(\Lambda\varphi, \tau)$ are given by the formulas (4.7) and (3.5), respectively.

Proof. Let (φ, ψ) be a solution of the problem (1.11)–(1.12). Take $\omega = [\frac{\varphi}{\psi'}]$. The formula (4.6) yields

$$A(\varphi)\omega_t + \omega' = \Psi(\varphi),$$

where

$$(4.30) \quad \Psi(\varphi) = \begin{bmatrix} \Lambda\varphi - \kappa b_2[u_2(t) + y_2(t)]q_1' \\ 0 \end{bmatrix}.$$

Set $\Upsilon = S\omega$. Then the estimate (4.14) is trivially true for $\Psi(\varphi)$ with $\Psi_1(\varphi) = 0$.

From the boundary condition $\varphi(L, t) = 0$ in (1.11) and the formula for S in (4.5), we have

$$(4.31) \quad \Upsilon_1(L, t) + \Upsilon_2(L, t) = \varphi(L, t) = 0, \quad t \geq 0.$$

Applying the inequalities (4.18) and (4.19) to $\Upsilon = (\Upsilon_1, \Upsilon_2)$ and using (4.31) gives

$$\begin{aligned} \int_0^t |\Upsilon(L, \nu)|^2 d\nu &= 2 \int_0^{T_{\xi_-, \xi_+}} \Upsilon_2^2(L, \nu) d\nu + 2 \int_{T_{\xi_-, \xi_+}}^t \Upsilon_1^2(L, \nu) d\nu \\ &\leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t [|\Upsilon|^2(0, \nu) + \Xi(\Upsilon)(\nu)] d\nu. \end{aligned}$$

Combining the above inequality and (4.16), we get for $2T_{\xi_-, \xi_+} \leq t \leq T$,

$$(4.32) \quad \int_0^t \|\Upsilon(\cdot, \nu)\|^2 d\nu \leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t [|\Upsilon(0, \nu)|^2 + \Xi(\Upsilon)(\nu)] d\nu.$$

On the other hand, from (3.13), (4.17), and (4.30), we obtain

$$(4.33) \quad \Xi(\Upsilon)(t) \leq C(\xi_-, \xi_+) \left[u_2^2(t) + y_2^2(t) + \|\Lambda\varphi(\cdot, t)\|^2 + \mathcal{E}^2(t) + \mathcal{E}^{\frac{3}{2}}(t) \right]$$

for all $t \in [0, T]$. In addition, it is easy to check that

$$C_1(\xi_-, \xi_+) [\varphi^2(x, t) + \psi'^2(x, t)] \leq |\Upsilon(x, t)|^2 \leq C_2(\xi_-, \xi_+) [\varphi^2(x, t) + \psi'^2(x, t)]$$

for all $(x, t) \in Q_T$. Using the inequalities (4.33), (4.32) and the boundary values in (3.15), where $v = \varphi$, $w = \psi$, we then obtain

$$\begin{aligned} (4.34) \quad &\int_0^t [\|\varphi(\cdot, \nu)\|^2 + \|\psi'(\cdot, \nu)\|^2] d\nu \\ &\leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \tilde{\mathcal{E}}_T^{\frac{1}{2}}} \int_0^t \left[|u(\nu)|^2 + |y(\nu)|^2 + \|\Lambda\varphi(\cdot, \nu)\|^2 + \mathcal{E}^2(\nu) + \mathcal{E}^{\frac{3}{2}}(\nu) \right] d\nu. \end{aligned}$$

Next, we take $\omega = [\frac{\varphi'}{\psi''}]$. We differentiate (4.6) with respect to x , obtaining

$$A(\varphi)\omega_t + \omega' = \Psi(\varphi),$$

where, this time,

$$\begin{aligned} \Psi(\varphi) &= \Psi_0(\varphi) - [A(\varphi)]' \vartheta_t, \\ \Psi_0(\varphi) &= \begin{bmatrix} (\Lambda\varphi)' - \kappa b_2[u_2(t) + y_2(t)]q_1'' \\ 0 \end{bmatrix}. \end{aligned}$$

In order to apply Lemma 4.3 to $\Upsilon = S(\varphi', \psi'')$ this time, we have to find $\Psi_1(\varphi)$ such that the estimate (4.14) is true, where $\Psi(\varphi)$ is as above. In fact, using the identity (4.8) and $|\Lambda\varphi| \leq C(\xi_-, \xi_+)|\nabla\varphi|_\infty$, we obtain (suppressing the variable t)

$$\begin{aligned} |\Psi(\varphi)| &\leq C(\xi_-, \xi_+) [|\Psi_0(\varphi)| + |\nabla\varphi|_\infty(|\varphi_t| + |\psi'_t|)] \\ &\leq C(\xi_-, \xi_+) [|\Psi_0(\varphi)| + |\nabla\varphi|_\infty|\Psi_1(\varphi)| + |\nabla\varphi|_\infty^2], \end{aligned}$$

where $\Psi_1(\varphi) = [u_2(t) + y_2(t)]q'_1$. Then a computation similar to that for (4.34) but this time with $\Upsilon = S(\varphi', \psi'')$ yields

$$\begin{aligned} &\int_0^t [\|\varphi'(\cdot, \nu)\|^2 + \|\psi''(\cdot, \nu)\|^2] d\nu \\ &\leq C(\xi_-, \xi_+) e^{c(\xi_-, \xi_+) \mathcal{E}_T^{\frac{1}{2}}} \int_0^t [\|(\Lambda\varphi)'(\cdot, \nu)\|^2 + |u|_1^2(\nu) + |y|_1^2(\nu) + \check{\mathcal{E}}^2(\nu) + \check{\mathcal{E}}^{\frac{3}{2}}(\nu)] d\nu. \end{aligned}$$

Similarly, when we apply Lemma 4.3 to $\Upsilon = S[\frac{\varphi''}{\psi'''}]$, $\Upsilon = S[\frac{\varphi_t}{\psi'_t}]$, and $\Upsilon = [\frac{\varphi_{tt}}{\psi'_{tt}}]$, respectively, we get the corresponding estimates for $\int_0^t (\|\varphi''\|^2 + \|\varphi'_t\|^2) d\nu$, $\int_0^t (\|\varphi_t\|^2 + \|\psi'_t\|^2) d\nu$, and $\int_0^t (\|\varphi_{tt}\|^2 + \|\psi'_{tt}\|^2) d\nu$, respectively. After we add all those inequalities together, we obtain (4.29). We omit the details here. $\square$

Remark 4.3. To prove Theorem 4.1, our next step is to absorb the lower order term $\int_0^t \mathbf{h}(\Lambda\varphi, \tau) d\tau$ in the inequality (4.29). Since the operator $\Lambda: L^2(Q_T) \rightarrow L^2(Q_T)$ (given by (4.7)) is compact, $\int_0^t \mathbf{h}(\Lambda\varphi, \tau) d\tau$ is a lower order term with respect to the energy level $\int_0^t \mathcal{E} d\tau$. This is shown by a compactness/uniqueness argument in the following lemma.

LEMMA 4.5. *Let $\xi_- \leq \varphi \leq \xi_+$ hold on $Q_{3T_{\xi_-, \xi_+}}$ and let $\varsigma > 0$ be given. Let T_{ξ_-, ξ_+} and $\check{\mathcal{E}}$ be defined by (4.1) and (4.2), respectively. Suppose that (φ_0, ψ_0) and $u \in [H^2(0, 3T_{\xi_-, \xi_+})]^2$ are such that the solution of the system (1.11)–(1.12) satisfies*

$$\check{\mathcal{E}}_{3T_{\xi_-, \xi_+}} \leq \varsigma.$$

Then there exists $C(\xi_-, \xi_+, \varsigma) > 0$, which may depend on ξ_-, ξ_+ , and ς but is independent of the initial data (φ_0, ψ_0) , such that

$$\begin{aligned} \int_0^{3T_{\xi_-, \xi_+}} \mathbf{h}(\Lambda\varphi, \nu) d\nu &\leq C(\xi_-, \xi_+, \varsigma) \left(\|u\|_{H^2(0, 3T_{\xi_-, \xi_+})}^2 + \|y\|_{H^2(0, 3T_{\xi_-, \xi_+})}^2 \right) \\ &\quad + C(\xi_-, \xi_+, \varsigma) \int_0^{3T_{\xi_-, \xi_+}} [\mathcal{E}^2(\nu) + \mathcal{E}^{\frac{3}{2}}(\nu)] d\nu, \end{aligned}$$

where $\mathbf{h}(\Lambda\varphi, \nu)$ is given by the formula (3.5).

Proof. By contradiction, assume that the conclusion of the lemma does not hold. Then, there exist initial data $(\varphi_{0,n}, \psi_{0,n})$, an input u_n , and the corresponding solutions (φ_n, ψ_n) of (1.11)–(1.12) over $(0, 3T_{\xi_-, \xi_+})$ such that, denoting $\tau = 3T_{\xi_-, \xi_+}$,

$$\xi_- \leq \varphi_n \leq \xi_+, \quad \check{\mathcal{E}}_{n, \tau} \leq \varsigma,$$

$$(4.35) \quad \frac{1}{n} \int_0^\tau \mathbf{h}(\Lambda\varphi_n, \nu) d\nu \geq \|u_n\|_{H^2(0, \tau)}^2 + \|y_n\|_{H^2(0, \tau)}^2 + \int_0^\tau [\mathcal{E}_n^2(\nu) + \mathcal{E}_n^{\frac{3}{2}}(\nu)] d\nu.$$

Since $\mathbf{h}(\Lambda\varphi_n, \nu) \leq C(\xi_-, \xi_+)\tilde{\mathcal{E}}_{n,\tau}$,

$$\|u_n\|_{H^2(0,\tau)}^2 + \|y_n\|_{H^2(0,\tau)}^2 + \int_0^\tau \left[\mathcal{E}_n^2(\nu) + \mathcal{E}_n^{\frac{3}{2}}(\nu) \right] d\nu \rightarrow 0,$$

which further implies

$$(4.36) \quad \|(\varphi_n, \psi_n)\|_{H^2(Q_\tau)}^2 + \int_0^\tau [\varphi_n'^2(0, \nu) + \|\psi_{tt}'(\cdot, \nu)\|^2] d\nu = \int_0^\tau \mathcal{E}_n(\nu) d\nu \rightarrow 0,$$

$$(4.37) \quad \sup_{(x,t) \in Q_\tau} |\varphi_n(x, t)| \rightarrow 0.$$

Set

$$c_n^2 = \int_0^\tau \mathbf{h}(\Lambda\varphi_n, \nu) d\nu, \quad \rho_n = \frac{\varphi_n}{c_n}, \quad \varrho_n = \frac{\psi_n}{c_n}, \quad \mu_n = \frac{u_n}{c_n}, \quad \nu_n = \frac{y_n}{c_n},$$

$$\mathcal{E}(\rho_n, \varrho_n)(t) = \rho_n'^2(0, t) + \|\varrho_{nt}'(\cdot, t)\|^2 + \sum_{|\alpha| \leq 2} [\|\partial^\alpha \rho_n(\cdot, t)\|^2 + \|\partial^\alpha \varrho_n(\cdot, t)\|^2].$$

Then

$$(4.38) \quad \|\Lambda\rho_n\|_{H^2(Q_\tau)}^2 + \int_0^\tau (\Lambda\rho_n')^2(0, t) dt = \int_0^\tau \mathbf{h}(\Lambda\rho_n, t) dt = 1,$$

and by (4.35)

$$(4.39) \quad \|\mu_n\|_{H^2(0,\tau)}^2 + \|\nu_n\|_{H^2(0,\tau)}^2 + \frac{1}{c_n^2} \int_0^\tau \left[\mathcal{E}_n^2(t) + \mathcal{E}_n^{\frac{3}{2}}(t) \right] dt \rightarrow 0.$$

We divide both sides of (4.29) by c_n^2 and use the relation (4.39) to see that the expressions below form a bounded sequence (indexed by n):

$$\|(\rho_n, \varrho_n)\|_{H^2(Q_\tau)}^2 + \int_0^\tau [\rho_n'^2(0, t) + \|\varrho_{nt}'(\cdot, t)\|^2] dt = \int_0^\tau \mathcal{E}(\rho_n, \varrho_n)(t) dt.$$

Since the operator Λ is compact, there are $\rho \in H^2(Q_\tau)$ and $\varrho \in H^{1,2}(Q_\tau) \cap H^2(Q_\tau)$ such that (by taking a subsequence)

$$(4.40) \quad \rho_n \rightarrow \rho \quad \text{weakly in } H^2(Q_\tau),$$

$$(4.41) \quad \Lambda\rho_n \rightarrow \Lambda\rho \quad \text{strongly in } H^2(Q_\tau),$$

$$(4.42) \quad \varrho_n \rightarrow \varrho \quad \text{weakly in } H^{1,2}(Q_\tau).$$

Moreover, it is easy to check that (ρ_n, ϱ_n) and φ_n satisfy for $(x, t) \in Q_\tau$

$$(4.43) \quad \begin{cases} \rho_{nt} = \mathcal{N}(\varphi_n)\varrho_n'', \\ \varrho_{nt} = \kappa\varrho_{nt}'' - \rho_n'', \\ \rho_n(L, t) = \varrho_n(L, t) = \varrho_n''(L, t) = 0, \\ \rho_n(0, t) = b_1[\mu_{n1}(t) + \nu_{1n}(t)], \\ \varrho_n'(0, t) = b_1^{-1}[\nu_{1n}(t) - \mu_{1n}(t)], \\ \rho_n'(0, t) - \kappa\varrho_{nt}'(0, t) = b_2[\mu_{2n}(t) + \nu_{2n}(t)], \\ \varrho_n(0, t) = b_2^{-1}[\mu_{n2}(t) - \nu_{2n}(t)]. \end{cases}$$

Therefore, the relations (4.36), (4.37), (4.39), and (4.40)–(4.43) imply that

$$(\rho, \varrho) \in H^2(Q_\tau) \times [H^{1,2}(Q_\tau) \cap H^2(Q_\tau)]$$

solves the following linear boundary-value problem for $(x, t) \in Q_\tau$:

$$(4.44) \quad \begin{cases} \rho_t = \mathcal{N}(0)\varrho'', \\ \varrho_t = \kappa\varrho_t'' - \rho'', \\ \rho(L, t) = \varrho(L, t) = \varrho''(L, t) = 0, \\ \rho(0, t) = \varrho(0, t) = 0, \\ \rho'(0, t) = \varrho'(0, t) = 0. \end{cases}$$

However, since $\tau > 2T_{\xi_-, \xi_+}$ it is easy to check that the linear problem (4.44) has only the zero solution, i.e., $(\rho, \varrho) = 0$ on Q_τ . This is a contradiction, since (4.38) and (4.41) imply $\int_0^\tau \mathbf{h}(\Lambda\rho, t)dt = 1$. $\square$

Remark 4.4. To guarantee that the problem (4.44) has only the zero solution, it is enough to assume that (4.44) holds on Q_{T_0} , where $T_0 > 2T_{\xi_-, \xi_+}$.

Proof of Theorem 4.1. It suffices to prove the inequality (4.3) for $s = 0$. We use a trick from Feng and Feng [5]. From Lemma 4.4 we see that (4.3) is true for $s = 0$ if we can find $C(\xi_-, \xi_+, \varsigma) > 0$ such that for $3T_{\xi_-, \xi_+} \leq t \leq T$

$$(4.45) \quad \int_0^t \mathbf{h}(\Lambda\varphi, \nu)d\nu \leq C(\xi_-, \xi_+, \varsigma) \left[\|u\|_{H^2(0,t)}^2 + \|y\|_{H^2(0,t)}^2 + \int_0^t \left[\mathcal{E}^2(\nu) + \mathcal{E}^{\frac{3}{2}}(\nu) \right] d\nu \right].$$

In fact, for any $h > 0$, from Lemma 4.5 we have, denoting again $\tau = 3T_{\xi_-, \xi_+}$,

$$(4.46) \quad \begin{aligned} \int_h^{\tau+h} \mathbf{h}(\Lambda\varphi, t)dt &= \int_0^\tau \mathbf{h}(\Lambda\varphi_h, t)dt \\ &\leq C(\xi_-, \xi_+, \varsigma) \left[\|u_h\|_{H^2(0,\tau)}^2 + \|y_h\|_{H^2(0,\tau)}^2 + \int_0^\tau \left[\mathcal{E}_h^2(t) + \mathcal{E}_h^{\frac{3}{2}}(t) \right] dt \right] \\ &= C(\xi_-, \xi_+, \varsigma) \left[\|u\|_{H^2(h,\tau+h)}^2 + \|y\|_{H^2(h,\tau+h)}^2 \right] \\ &\quad + C(\xi_-, \xi_+, \varsigma) \int_h^{\tau+h} \left[\mathcal{E}^2(t) + \mathcal{E}^{\frac{3}{2}}(t) \right] dt, \end{aligned}$$

where $\varphi_h(t) = \varphi(t+h)$, $u_h(t) = u(t+h)$, $y_h(t) = y(t+h)$, and $\mathcal{E}_h$ is defined by (1.18) with (φ_h, ψ_h) in place of (φ, ψ) . Denote the right-hand side of (4.46) by $I(h, \tau+h)$. Then for $m\tau \leq t \leq (m+1)\tau$ with $m \geq 2$ an integer, we obtain

$$\begin{aligned} \int_0^t \mathbf{h}(\Lambda\varphi, \nu)d\nu &\leq \int_0^\tau \mathbf{h}(\Lambda\varphi, \nu)d\nu + \cdots + \int_{(m-1)\tau}^{m\tau} \mathbf{h}(\Lambda\varphi, \nu)d\nu + \int_{t-\tau}^t \mathbf{h}(\Lambda\varphi, \nu)d\nu \\ &\leq I(0, \tau) + \cdots + I((m-1)\tau, m\tau) + I(t-\tau, t) \leq 2I(0, t), \end{aligned}$$

which is just the inequality (4.45). $\square$

From the definitions (1.18) and (3.9), it is easy to check that if the solution (φ, ψ) of the system (1.11)–(1.12) satisfies $\xi_- \leq \varphi \leq \xi_+$, then there are $C_1(\xi_-, \xi_+) > 0$ and $C(\xi_-, \xi_+) > 0$ such that

$$(4.47) \quad C_1(\xi_-, \xi_+)\mathcal{E}(t) \leq \Pi(t) \leq C(\xi_-, \xi_+)\mathcal{E}(t), \quad 0 \leq t \leq T.$$

For convenience, we rewrite Theorem 4.1 as follows.

THEOREM 4.1*. Suppose that $\sigma_- < \xi_- < 0 < \xi_+ < \sigma_+$, $\varsigma > 0$, and $T > 3T_{\xi_-, \xi_+}$. Then there exists a number $C(\xi_-, \xi_+, \varsigma) > 0$, which depends on ξ_-, ξ_+ and ς , such that if the solution (φ, ψ) of the system (1.11)–(1.12) satisfies the assumptions

$$\xi_- \leq \varphi \leq \xi_+, \quad \sup_{0 \leq t \leq T} \Pi(t) \leq \varsigma,$$

then for $0 \leq s \leq t \leq T$ with $t - s \geq 3T_{\xi_-, \xi_+}$,

$$\int_s^t \Pi(\nu) d\nu \leq C(\xi_-, \xi_+, \varsigma) \left[\|u\|_{H^2(s, t)}^2 + \|y\|_{H^2(s, t)}^2 + \int_s^t \left[\Pi^2(\nu) + \Pi^{\frac{3}{2}}(\nu) \right] d\nu \right].$$

5. Global smooth solutions. In this section we establish the existence of the global smooth solutions and the exponential decay of the Sobolev norm for small initial data.

Let $u(t) = (u_1, u_2) \in H^2(0, \infty) \times H^2(0, \infty)$. Suppose that $(\varphi_0, \psi_0) \in H^2(0, L) \times H^3(0, L)$ satisfies the boundary conditions (1.15) and the compatibility conditions (1.16). We assume that (φ, ψ) solves the problem (1.11)–(1.12) on $Q_T = (0, L) \times (0, T)$ for some $T > 0$ such that the regularity (1.17) holds.

Throughout this section we fix ξ such that

$$\sigma_- < -\xi < 0 < \xi < \sigma_+.$$

We introduce some constants that we need later: Let $C(-\xi, \xi) > 0$ be given by Theorem 3.1*, (4.47), and (3.29) for $\xi_- = -\xi$ and $\xi_+ = \xi$. Set

$$\varsigma = \min \left\{ 1, \frac{1}{C(-\xi, \xi)}, \frac{\xi^2}{2LC(-\xi, \xi)} \right\}.$$

Let $C(-\xi, \xi, \varsigma)$ be given by Theorem 4.1* for $\xi_- = \xi$, $\xi_+ = \xi$, and the above ς . We then fix $\gamma > 0$ small such that

$$(5.1) \quad \varpi(\gamma) = C^{-1}(-\xi, \xi, \varsigma) - 2[2\gamma + \gamma^{\frac{1}{2}}] > 0,$$

$$(5.2) \quad c(\gamma) = \varpi - 2C(-\xi, \xi)[2\gamma + \gamma^{\frac{1}{2}}] > 0,$$

$$(5.3) \quad T_0(\gamma) = \frac{1}{2C(-\xi, \xi)\gamma^{\frac{1}{2}}} > 3T_{-\xi, \xi},$$

where $T_{-\xi, \xi}$ is given by (4.11) for $\xi_- = \xi$ and $\xi_+ = \xi$.

Proof of Theorem 1.2. We fix

$$(5.4) \quad 0 < \gamma < \varsigma/4$$

such that the inequalities (5.1)–(5.3) hold.

We assume that the initial data (φ_0, ψ_0) and the input u are such that

$$(5.5) \quad \mathcal{E}(0) + \|u\|_{H^2(0, \infty)}^2 < \min \left\{ \frac{\xi^2}{2L}, \frac{\gamma}{C(-\xi, \xi) + 1} \right\},$$

where $\mathcal{E}(t)$ is given by (1.18).

Step I. We prove that, under the small initial condition (5.5), the problem (1.11)–(1.12) has global solutions. The condition (5.5) yields

$$(5.6) \quad \sup_{x \in [0, L]} |\varphi_0(x)| \leq L^{\frac{1}{2}} \mathcal{E}^{\frac{1}{2}}(0) < \frac{\xi}{\sqrt{2}},$$

which, by Theorem 1.1, implies that the short-time solution (φ, ψ) of the problem (1.11)–(1.12) exists. For $0 \leq s \leq t$, set

$$(5.7) \quad \begin{aligned} h(s, t) &= \Pi(t) + \|y\|_{H^2(s, t)}^2, \\ g(s, t) &= \Pi(s) + C(-\xi, \xi) \|u\|_{H^2(s, t)}^2. \end{aligned}$$

By (4.47), (5.5), and (5.4), we have

$$(5.8) \quad g(0, \infty) \leq C(-\xi, \xi) [\mathcal{E}(0) + \|u\|_{H^2(0, \infty)}^2] \leq \gamma < \varsigma/4.$$

In particular, $\Pi(0) \leq \varsigma/4$. This inequality together with (5.6) implies that the local solution must satisfy

$$(5.9) \quad -\xi \leq \varphi \leq \xi, \quad x \in [0, L], \quad \Pi(t) \leq \varsigma,$$

for t in some interval $[0, \delta]$.

Let δ_0 be the largest number such that the estimates in (5.9) hold for $t \in [0, \delta_0]$. We shall prove that $\delta_0 = \infty$ by contradiction. Then the problem (1.11)–(1.12) has global solutions under the small initial data condition (5.5).

Let $\delta_0 < \infty$. Then (3.36) implies that for $t \in [0, \min\{\delta_0, T_0(\gamma)\}]$

$$(5.10) \quad \begin{aligned} h(0, t) &\leq g(0, t) + C(-\xi, \xi) \int_0^t \Pi^{\frac{3}{2}}(\Pi^{\frac{1}{2}} + 1) dt \\ &\leq g(0, \min\{\delta_0, T_0(\gamma)\}) + 2C(-\xi, \xi) \int_0^t h^{\frac{3}{2}}(0, t) dt, \end{aligned}$$

since $g(0, \cdot)$ is increasing. By (5.8), we have

$$(5.11) \quad \min\{\delta_0, T_0(\gamma)\} \leq T_0(\gamma) \leq \frac{1}{2C(-\xi, \xi)g^{\frac{1}{2}}(0, \min\{\delta_0, T_0(\gamma)\})}.$$

We obtain from (5.8), (5.10), and (5.11) that

$$(5.12) \quad \begin{aligned} \Pi(t) + \|y\|_{H^2(0, t)}^2 &\leq g(0, \min\{\delta_0, T_0(\gamma)\}) / [1 - tC(-\xi, \xi)g^{\frac{1}{2}}(0, \min\{\delta_0, T_0(\gamma)\})]^2 \\ &\leq 4g(0, \min\{\delta_0, T_0(\gamma)\}) \leq 4\gamma < \varsigma \end{aligned}$$

for $0 \leq t \leq \min\{\delta_0, T_0(\gamma)\}$.

We assume that $T_0(\gamma) \geq \delta_0$. Then the inequality (5.12) shows that $\Pi(t) \leq 4\gamma < \varsigma$ holds for $0 \leq t \leq \delta_0$, which gives, via (3.29),

$$\begin{aligned} \sup_{0 \leq x \leq L} |\varphi(x, t)| &\leq L^{\frac{1}{2}} \mathbf{h}^{\frac{1}{2}}(\varphi, t) \leq [LC(-\xi, \xi)\Pi(t)]^{\frac{1}{2}} \\ &\leq 2[LC(-\xi, \xi)]^{\frac{1}{2}} \gamma^{\frac{1}{2}} \leq \frac{\xi}{\sqrt{2}}, \quad 0 \leq t \leq \delta_0. \end{aligned}$$

Then the conditions (5.9) are true for $t \in [0, \delta_1)$ with $\delta_1 > \delta_0$. This is a contradiction. Thus, we obtain that $T_0(\gamma) < \delta_0$.

Now the inequality (5.12) implies

$$(5.13) \quad \Pi(t) \leq 4\gamma, \quad 0 \leq t \leq T_0(\gamma).$$

We apply Theorem 4.1* to give for $T_0(\gamma) \geq t \geq 3T_{-\xi, \xi}$

$$(5.14) \quad \varpi \int_0^t \Pi(\tau) d\tau \leq \|u\|_{H^2(0,t)}^2 + \|y\|_{H^2(0,t)}^2.$$

We then add the inequalities (3.36) with $\xi_- = -\xi$, $\xi_+ = \xi$, and (5.14) to deduce from (5.13) that, for $3T_{-\xi, \xi} \leq t \leq T_0(\gamma)$,

$$(5.15) \quad \begin{aligned} & \Pi(t) + \varpi \int_0^t \Pi(\tau) d\tau + \|y\|_{H^2(0,t)}^2 \\ & \leq \Pi(0) + [C(-\xi, \xi) + 1] \|u\|_{H^2(0,t)}^2 + \|y\|_{H^2(0,t)}^2 \\ & \quad + 2C(-\xi, \xi)[2\gamma + \gamma^{\frac{1}{2}}] \int_0^t \Pi(\tau) d\tau, \end{aligned}$$

that is,

$$(5.16) \quad \Pi(t) + c(\gamma) \int_0^t \Pi(\tau) d\tau \leq \Pi(0) + [C(-\xi, \xi) + 1] \|u\|_{H^2(0,t)}^2,$$

for $3T_{-\xi, \xi} \leq t \leq T_0(\gamma)$. From (5.7), (5.16), (4.47), and (5.5) we have

$$\begin{aligned} g(T_0(\gamma), \infty) & \leq \Pi(0) + [C(-\xi, \xi) + 1] \|u\|_{H^2(0, T_0(\gamma))}^2 + C(-\xi, \xi) \|u\|_{H^2(T_0(\gamma), \infty)}^2 \\ & \leq \Pi(0) + [C(-\xi, \xi) + 1] \|u\|_{H^2(0, \infty)}^2 \\ & \leq C(-\xi, \xi) \mathcal{E}(0) + [C(-\xi, \xi) + 1] \|u\|_{H^2(0, \infty)}^2 \leq \gamma. \end{aligned}$$

Next, we restart the above procedure from $g(T_0(\gamma), \infty) < \gamma$ and using $T_0(\gamma) \leq t \leq 2T_0(\gamma)$ instead of $0 \leq t \leq T_0(\gamma)$ and we arrive at

$$2T_0(\gamma) < \delta_0.$$

If we continue, by induction we obtain that for any $k \in \mathbb{N}$,

$$kT_0(\gamma) < \delta_0.$$

This is also a contradiction.

Step II. By Step I, under the condition (5.5), the estimates (5.9) hold for all $t \in [0, \infty)$. In the interval $[0, \infty)$ we apply Theorems 3.1* and 4.1* and, by the extension from $[iT_0(\gamma), (i+1)T_0(\gamma)]$ to $[(i+1)T_0(\gamma), (i+2)T_0(\gamma)]$ for $i \geq 0$, obtain that the inequality (5.12) holds for all $t \in [0, \infty)$. Then, an argument similar to that for (5.16) yields that for $t \in [3T_{-\xi, \xi}, \infty)$,

$$\Pi(t) + c(\gamma) \int_0^t \Pi(\tau) d\tau \leq \Pi(0) + [C(-\xi, \xi) + 1] \|u\|_{H^2(0,t)}^2.$$

By (4.47), the proof is complete. $\square$

Proof of Theorem 1.3. Let the input u satisfy

$$u(t) = 0 \quad \forall t \geq T_0.$$

We repeat the argument in the proof of Theorem 1.2 for $s \geq t \geq T_0$ to obtain that for $s \geq t \geq T_0$,

$$\Pi(s) + c(\gamma) \int_t^s \Pi(\tau) d\tau \leq \Pi(t) + [C(-\xi, \xi) + 1] \|u\|_{H^2(t, \infty)}^2 = \Pi(t),$$

which yields

$$\Pi(t) \leq \Pi(T_0) e^{-c(\gamma)(t-T_0)} \quad \forall t \geq T_0.$$

The decay rate estimate (1.24) is true because of the inequality (4.47). $\square$

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